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SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS WITH POLYNOMIAL SOLUTIONS

By T. W. CHAUNDY (*Oxford*)

[Received 28 January 1952; in revised form 9 January 1953]

1. Introduction

THE purpose of this note is to amplify and correct an earlier note of mine (1) with a similar title in this Journal.† W. W. Sawyer (2) had considered the problem of finding differential operators F, G such that the system of differential equations

$$(F-\lambda G)Y = 0 \tag{1}$$

should have polynomial solutions of every order n (zero included) when, for each n, we give λ its appropriate value λ_n.

To this problem I supposed that I had given a sufficient answer until Professor Sawyer pointed out two major defects in my argument. The general picture remains true, I think, but these details of the sketch were overhasty or false.

Here I shall limit myself to Sawyer's original statement of the problem: that is, to equations of the second order only. This, of course, simplifies the discussion, but I trust that the method and arguments can be enlarged to deal with equations of higher order.

2. Reducibility

2.1. As before, I express the equation (1) in the form

$$[F(x,\delta)-\lambda G(x,\delta)]Y = 0, \tag{2}$$

where δ is the operator xd/dx, and

$$F(x,\delta) = \sum_{r=0}^{p} x^r f_r(\delta), \qquad G(x,\delta) = \sum_{r=0}^{p} x^r g_r(\delta), \tag{3}$$

these operators being of rank p. Replacing λ by λ_n we have the identity

$$F(x,\delta)Y_n = \lambda_n G(x,\delta)Y_n \tag{4}$$

defining the polynomial solution

$$Y_n = x^n + a_{n,n-1}x^{n-1} + \ldots + a_{n0}. \tag{5}$$

† By a strange editorial oversight that note was ante-dated by a year.

Quart. J. Math. Oxford (2), 4 (1953), 81–95.

From (4) we deduce [(**1**) (9), (10)] the 'derived identity'

$$F(x,\delta)x^n = G(x,\delta)Z_n, \tag{6}$$

where $$Z_n = \lambda_n x^n + b_{n,n-1}x^{n-1} + \ldots + b_{n0}.$$

We are always able [(**1**) § 2.1] to put (2) into a form in which $0, \infty$ are not values of λ_n. We can thus always interchange F and G by writing λ^{-1} for λ. The asymmetry in (6) is therefore not intrinsic but from choice, and we can obtain a second 'derived identity' of the form

$$F(x,\delta)Z'_n = G(x,\delta)x^n. \tag{7}$$

2.2. It is convenient to speak of a system (2) which has polynomial solutions of every degree as a *Sawyer system*. It may happen that the system has polynomial solutions of all degrees only from some m upward. I shall call this an *incomplete* Sawyer system (of degree m). As we saw in (**1**), if $[F(x,\delta)-\lambda G(x,\delta)]Y = 0$ is a Sawyer system, then

$$[F(x,\delta-m)-\lambda G(x,\delta-m)]Y = 0$$

is an incomplete Sawyer system of degree m.

Now a Sawyer system is defined only through certain of its solutions, and the equation itself is therefore imprecise. Thus, if (2) is a Sawyer system, so is any

$$\Theta(x,\delta)[F(x,\delta)-\lambda G(x,\delta)]Y = 0.$$

We must therefore be ready to remove any such irrelevant factor as $\Theta(x,\delta)$, which can falsify both rank and order.

This reducibility is effected, as in (**1**), by considering the solutions in series of the differential equation

$$G(x,\delta)y = 0. \tag{8}$$

My previous arguments, however, were unsatisfactory and I reconsider the whole matter here.

2.3. It is clear from (6), especially if we think of n as 'large', that Z_n is part of a series solution of (8). Precisely, Z_n will be such a solution correct to x^{n-1} since x^n is the lowest power of x on the left of (6). By 'series solution' one must mean, by the conditions of the problem, a series of non-negative integer powers of x, and I shall use the term in this precise sense throughout.

The leading term x^m of this series is given by the indicial equation

$$g_0(\delta)x^m = 0, \tag{9}$$

where we recall the form of $G(x,\delta)$ stated in (3). Now in the limited form of the problem every f, g is at most of the second order, and thus

this indicial equation (9) has at most two integral solutions. There are thus three possibilities according as (8) has 0, 1, 2 different series solutions.

2.4. The case of no series solutions was adequately discussed in § 4.5 of (1). Then Z_n is always $\lambda_n x^n$ (in fact $Z_n = \lambda_n Y_n$) and necessarily

$$\lambda_n = \frac{f_0(n)}{g_0(n)} = \ldots = \frac{f_p(n)}{g_p(n)}.$$

If these equal fractions, reduced to lowest terms, are $f(n)/g(n)$, then $F(x,\delta)-\lambda G(x,\delta)$ is merely some

$$\Theta(x,\delta)[f(\delta)-\lambda g(\delta)],$$

and the relevant solutions are those of $[f(\delta)-\lambda g(\delta)]Y = 0$. In this case, then, the system has been reduced to a system of rank zero.

3. The hypergeometric case

3.1. I come now to the case in which (8) has just one series solution; and I shall prove that then the system is reducible to the hypergeometric form (in the narrower sense), i.e. to a system of rank one. In order to have anything to prove we therefore suppose at the outset that $p \geqslant 2$. Let the single series solution be

$$y = e_0 x^m + e_1 x^{m+1} + e_2 x^{m+2} + \ldots . \tag{10}$$

We should notice, incidentally, that this must be an infinite series. For, if it were the terminated series $e_0 x^m + \ldots + e_r x^{m+r}$, then taking $n > m+r$, we must have, for some constant A_n,

$$F(x,\delta)x^n = G(x,\delta)[A_n(e_0 x^m + \ldots + e_r x^{m+r}) + \lambda_n x^n].$$

Hence

$$F(x,\delta)x^n = \lambda_n G(x,\delta)x^n,$$

and this for all $n > m+r$. This reproduces the situation of § 2.4 immediately preceding, so that in such a case the system would be reducible to a system of rank zero.

Returning to (10) we have the derived identity in the form

$$F(x,\delta)x^n = G(x,\delta)[A_n(e_0 x^m + \ldots + e_{n-m-1} x^{n-1}) + \lambda_n x^n], \tag{11}$$

where A_n is some constant depending on n. This presumes, of course, that $n > m$. When $n \leqslant m$, we must still have the form of § 2.4,

$$F(x,\delta)x^n = \lambda_n G(x,\delta)x^n. \tag{12}$$

Putting $r = n-m$, we can rewrite (11) as

$$F(x,\delta+m)x^r = G(x,\delta+m)[A_n(e_0+e_1 x+\ldots+e_{r-1} x^{r-1}) + \lambda_n x^r]. \tag{13}$$

This is the derived identity for the system

$$F(x,\delta+m)-\lambda G(x,\delta+m). \tag{14}$$

It holds for all positive r; the case $r = 0$ is covered by $n = m$ in (12). Thus the derived identity is true also for (14), and therefore, by (1) § 2.4, so is the defining identity. Thus (14) is also a Sawyer system; and to prove it hypergeometric is also to prove the original system hypergeometric.

In other words we may sufficiently suppose that the series solution begins with a constant term as it does in (13), so that now (11) is replaced by

$$F(x,\delta)x^n = G(x,\delta)[A_n(e_0+\ldots+e_{n-1}x^{n-1})+\lambda_n x^n]. \tag{15}$$

3.2. I now show that this series is itself hypergeometric, i.e. that it satisfies a differential equation of rank one, or equivalently that the coefficients e_n satisfy a two-term recurrence formula.

Written out, the derived identity (12) is

$$\begin{aligned}&[f_0(\delta)+\ldots+x^p f_p(\delta)]x^n\\ &\quad= [g_0(\delta)+\ldots+x^p g_p(\delta)]\{A_n(e_0+\ldots+e_{n-1}x^{n-1})+\lambda_n x^n\}\\ &\quad= [g_0(\delta)+\ldots+x^p g_p(\delta)]\{\lambda_n x^n - A_n(e_n x^n+e_{n+1}x^{n+1}+\ldots \text{ to } \infty)\}\end{aligned}$$

since

$$G(x,\delta)(e_0+e_1 x+\ldots+ \text{ to } \infty) = 0.$$

Identification of coefficients here provides redundance of information. Profiting by a suggestion of Professor Burchnall's let us identify the coefficients of x^{n+p-1}, x^{n+p} in the first and second members of these equations and the coefficients of x^n, x^{n+1} in the first and third members. We get

$$f_{p-1}(n)-\lambda_n g_{p-1}(n) = A_n e_{n-1} g_p(n-1), \tag{16}$$

$$f_p(n)-\lambda_n g_p(n) = 0, \tag{17}$$

$$f_0(n)-\lambda_n g_0(n) = -A_n e_n g_0(n), \tag{18}$$

$$f_1(n)-\lambda_n g_1(n) = -A_n\{e_n g_1(n)+e_{n+1}g_0(n+1)\}. \tag{19}$$

Elimination of λ_n, A_n between (17), (18), (19) gives on reduction

$$\frac{e_{n+1}}{e_n} = \frac{g_p(n)\{f_0(n)g_1(n)-f_1(n)g_0(n)\}}{g_0(n+1)\{f_p(n)g_0(n)-f_0(n)g_p(n)\}}. \tag{20}$$

Similarly, elimination of λ_n, A_n between (16), (17), (18) gives

$$\frac{e_n}{e_{n-1}} = \frac{g_p(n-1)\{f_0(n)g_p(n)-f_p(n)g_0(n)\}}{g_0(n)\{f_{p-1}(n)g_p(n)-f_p(n)g_{p-1}(n)\}}, \tag{20a}$$

a form alternative to (20). Write briefly

$$U_{rs} \equiv f_r g_s - f_s g_r.$$

Then the series $y = \sum e_n x^n$ satisfies an equation of rank one which may be written in either of the forms

$$\left.\begin{aligned} g_0(\delta)U_{p0}(\delta-1)y &= x\,g_p(\delta)U_{01}(\delta)y \\ g_0(\delta)U_{p-1,p}(\delta)y &= x\,g_p(\delta)U_{0p}(\delta)y \end{aligned}\right\}. \tag{21}$$

This indicates an identity between the f_r, g_s which it would not be profitable to explore; there are clearly several such.

When, in particular we restrict F, G, and therefore f, g, to be of order two, it appears that the differential equation (21) is at most of order six.

3.3. The preceding argument shows that F and so, by symmetry, G as well, are of rank one, at least when possible left-hand factors have been removed. But this argument is not exact enough to prove that $F-\lambda G$ itself is effectively of rank one.

If, in (15), we replace n by r and substitute for every $F(x,\delta)x^r$ in the defining identity

$$F(x,\delta)\sum_{r=0}^{n} a_{nr}x^r = \lambda_n\, G(x,\delta)\sum_{r=0}^{n} a_{nr}x^r,$$

we get

$$G(x,\delta)\sum_{r=0}^{n} a_{nr}\{A_r(e_0+\ldots+e_{r-1}x^{r-1})+(\lambda_r-\lambda_n)x^r\} = 0. \tag{22}$$

It was seen in § 3.1 that the single series solution of $G(x,\delta)y = 0$ is an infinite series. Thus the operand in (22) must vanish identically and, from the coefficient of x^r, we have

$$(\lambda_r-\lambda_n)a_{nr}+e_r\sum_{s=r+1}^{n} a_{ns}A_s = 0,$$

and so

$$\frac{\lambda_n-\lambda_r}{e_r}a_{nr}-\frac{\lambda_n-\lambda_{r+1}}{e_{r+1}}a_{n,r+1} = a_{n,r+1}A_{r+1}.$$

This gives the recurrence formula

$$\frac{a_{n,r+1}}{a_{nr}} = \frac{e_{r+1}(\lambda_n-\lambda_r)}{e_r(\lambda_n-\lambda_{r+1}+e_{r+1}A_{r+1})}. \tag{23}$$

Write $r+1$ for n in (18) and r for n in (17) and substitute for $\lambda_{r+1}-A_{r+1}e_{r+1}$ and λ_r from these equations into (23). We get

$$\frac{a_{n,r+1}}{a_{nr}} = \frac{e_{r+1}g_0(r+1)\{f_p(r)-\lambda_n g_p(r)\}}{e_r g_p(r)\{f_0(r+1)-\lambda_n g_0(r+1)\}}.$$

Finally, substitution for e_{r+1}/e_r from (20) with r replacing n gives

$$\frac{a_{n,r+1}}{a_{nr}} = \frac{\{f_p(r)-\lambda_n g_p(r)\}U_{01}(r)}{\{f_0(r+1)-\lambda_n g_0(r+1)\}U_{p0}(r)}.$$

Thus
$$Y_n = \sum_{r=0}^{n} a_{nr} x^r$$
satisfies the differential equation
$$\{f_0(\delta)-\lambda_n g_0(\delta)\}U_{p0}(\delta-1)Y_n = x\{f_p(\delta)-\lambda_n g_p(\delta)\}U_{01}(\delta)Y_n,$$
which we may rearrange as
$$[f_0(\delta)U_{p0}(\delta-1)-xf_p(\delta)U_{01}(\delta)]Y_n = \lambda_n[g_0(\delta)U_{p0}(\delta-1)-xg_p(\delta)U_{01}(\delta)]Y_n. \tag{24}$$
This is exactly equivalent to
$$F(x,\delta)Y_n = \lambda_n G(x,\delta)Y_n$$
and so either is identical with it or may replace it. Thus, when $G(x,\delta)$ has a single series solution, the system is reducible to the form (24).

This form and the analysis leading to it I owe to Professor Burchnall; it replaces a clumsy result of my own.

3.4. In full the equation to the system is
$$\begin{aligned}[f_0(\delta)\{f_p(\delta-1)g_0(\delta-1)-f_0(\delta-1)g_p(\delta-1)\}-& \\ -xf_p(\delta)\{f_0(\delta)g_1(\delta)-f_1(\delta)g_0(\delta)\}]Y& \\ = \lambda[g_0(\delta)\{f_p(\delta-1)g_0(\delta-1)-f_0(\delta-1)g_p(\delta-1)\}-& \\ -xg_p(\delta)\{f_0(\delta)g_1(\delta)-f_1(\delta)g_0(\delta)\}]Y.&\end{aligned} \tag{25}$$
In the particular case in which F, G are of order two, this equation is of rank one and order six. It has the same polynomial solutions as the original system, which is of rank p ($\geqslant 2$) and order two. Thus to reduce the rank we have had to accept an increase in the order. This is a phenomenon to which Burchnall has elsewhere called attention.

4. Equations of rank two

4.1. *Two series solutions.* I come now to the case in which $G(x,\delta)y = 0$ has two distinct solutions in *infinite* series; as in § 3.1, terminated solutions do not count. Since we have already dealt with the hypergeometric case, we sufficiently suppose the system to be of rank two or more. I begin with the case in which the rank is exactly two.

Let us first suppose that the series solutions are S_0, S_m with leading indices 0, m, so that sufficiently
$$g_0(\delta) \equiv \delta(\delta-m). \tag{26}$$
A homographic change on λ, say
$$\lambda \to (h\lambda-h')/(\lambda-k'),$$

puts the system into the form

$$(k'F-h'G)Y = \lambda(F-hG)Y. \tag{27}$$

Here h, h', k' are arbitrary except that h', k' are so chosen that $0, \infty$ are still not values of λ_n. Then again, unless the system is to be hypergeometric, $(F-hG)y = 0$ must have two distinct series solutions. Thus the indicial equation

$$f_0(t)-h\,g_0(t) = 0 \tag{28}$$

must have two non-negative integer solutions. When h is not a member of the infinite sequence $f_0(n)/g_0(n)$ $(n = 0, 1,...)$, (28) has no positive integer or zero solution except possibly at common zeros of f_0, g_0. Since there are to be two such solutions, f_0, g_0 must have the same factors, so that sufficiently we may write

$$f_0(\delta) \equiv \delta(\delta-m) \equiv g_0(\delta). \tag{29}$$

4.2. As we know, in the ordinary theory of solutions in series, the presence in the indicial equation of two roots differing by an integer may lead to a logarithmic solution. The need to avoid this imposes a further condition on G. It is that the recurrence relations determining the coefficients in the series S_0 leave indeterminate the coefficient of x^m. This is clearly necessary, for we change the coefficient of x^m when (as we may) we consider any S_0+CS_m in place of S_0.

For a simple example take $m = 3$ and write

$$S_0 = e_0+e_1x+e_2x^2+e_3x^3+\dots.$$

Then

$$[g_0(\delta)+x\,g_1(\delta)+x^2g_2(\delta)]S_0 = 0$$

gives from the coefficients of x, x^2, x^3, since $g_0(3) = 0$,

$$\left.\begin{aligned} e_0g_1(0)+e_1g_0(1) &= 0\\ e_0g_2(0)+e_1g_1(1)+e_2g_0(2) &= 0\\ e_1g_2(1)+e_2g_1(2) &= 0 \end{aligned}\right\}, \tag{30}$$

and hence necessarily

$$\begin{vmatrix} g_1(0) & g_0(1) & 0\\ g_2(0) & g_1(1) & g_0(2)\\ 0 & g_2(1) & g_1(2) \end{vmatrix} = 0. \tag{31}$$

A homographic transformation on λ, as in § 4.1 immediately above, extends this to the wider necessary condition

$$\begin{vmatrix} f_1(0)-h\,g_1(0) & f_0(1)-h\,g_0(1) & 0\\ f_2(0)-h\,g_2(0) & f_1(1)-h\,g_1(1) & f_0(2)-h\,g_0(2)\\ 0 & f_2(1)-h\,g_2(1) & f_1(2)-h\,g_1(2) \end{vmatrix} = 0. \tag{32}$$

This holding for arbitrary h is equivalent to four independent conditions.

For general m the corresponding determinant is of order m and represents $m+1$ independent conditions. The rth row consists of

$$f_2(r-2)-h\,g_2(r-2),\quad f_1(r-1)-h\,g_1(r-1),\quad f_0(r)-h\,g_0(r) \qquad (33)$$

preceded by $r-2$ zeros and followed by $m-r-1$ zeros. The first and last rows in the determinant are truncated in the way that can be seen in (32) above.

5. Y_n exists when $n > m$

Now it is readily seen that, independently of the foregoing conditions, the derived identity can be satisfied for all n greater than m. Write $S_{0(r)}$, $S_{m(r)}$ for the sum in S_0, S_m up to and including x^r. Then the derived identity has to be

$$[f_0(\delta)+x f_1(\delta)+x^2 f_2(\delta)]x^n$$
$$=[g_0(\delta)+x\,g_1(\delta)+x^2 g_2(\delta)]\{\lambda_n x^n+A_n S_{0(n-1)}+B_n S_{m(n-1)}\}, \qquad (34)$$

for suitable constants A_n, B_n. But, since S_0 is a series solution of $G(x,\delta)y=0$, the technique of such solutions tells us that $G(x,\delta)S_{0(n-1)}$ has a form $k_n x^n+k_{n+1}x^{n+1}$ for some k_n, k_{n+1}; and so for $S_{m(n-1)}$. Thus (34) becomes

$$f_0(n)x^n+f_1(n)x^{n+1}+f_2(n)x^{n+2}$$
$$=\lambda_n\{g_0(n)x^n+g_1(n)x^{n+1}+g_2(n)x^{n+2}\}+$$
$$+A_n(k_n x^n+k_{n+1}x^{n+1})+B_n(k'_n x^n+k'_{n+1}x^{n+1}).$$

This is a three-term relation and is satisfied by giving λ_n its usual value $f_2(n)/g_2(n)$ and by suitable choice of A_n, B_n.

This determination would be impossible if

$$\begin{vmatrix} k_n & k_{n+1} \\ k'_n & k'_{n+1} \end{vmatrix} = 0.$$

But then we could find a linear combination of S_0, S_m lacking the terms x^n, x^{n+1}, and therefore every subsequent term since the S are annihilated by a three-term operator. In this case the two solutions are in effect one infinite and one terminated, a position excluded by hypothesis. Thus (34) is satisfied for all $n > m$: to secure the presence of $S_{m(n-1)}$ we have to exclude $n = m$.

6. Existence of Y_n when $n \leqslant m$

6.1. To complete the proof that $F-\lambda G$ is a Sawyer system we need to establish the derived identity when $n \leqslant m$. It is easier to work with the polynomials Y_n. If they exist when $n \leqslant m$, so do the Z_n: in that case the Z_n will exist for all n and, arguing back, we have the Y_n for all n.

Counting tells us that we need just $m+1$ conditions to permit the existence of these $m+1$ solutions Y_n; and this $m+1$ is the number given by the vanishing of (32) for all h. In point of fact the conditions given by (32) can be shown to be necessary but not, I think, sufficient for the existence of the Y_n ($n \leqslant m$).

For instance, the existence of a polynomial solution Y_1 with $\lambda = \lambda_1$ requires the existence of the identity (with some a_0)

$$[\{f_0(\delta)-\lambda_1 g_0(\delta)\}+x\{f_1(\delta)-\lambda_1 g_1(\delta)\}+x^2\{f_2(\delta)-\lambda_1 g_2(\delta)\}](a_0+x) = 0.$$

In the expansion of this the constant term disappears because of the form of $f_0(\delta)$, $g_0(\delta)$; the term in x^3 disappears, since, by the general form of λ_n,

$$f_2(1)-\lambda_1 g_2(1) = 0. \tag{35}$$

The vanishing of the coefficients of x, x^2 gives on elimination of a_0 the vanishing of the determinant

$$\begin{vmatrix} f_1(0)-\lambda_1 g_1(0) & f_0(1)-\lambda_1 g_0(1) \\ f_2(0)-\lambda_1 g_2(0) & f_1(1)-\lambda_1 g_1(1) \end{vmatrix}. \tag{36}$$

If in (32) we put $h = \lambda_1$ and use (35), the determinant breaks up into the product of two determinants, one of which is (36). Thus the existence of Y_1 gives $h = \lambda_1$ as a root of (32). The existence of Y_0 gives similarly the vanishing of

$$f_1(0)-\lambda_0 g_1(0) \tag{37}$$

together with

$$f_2(0)-\lambda_0 g_2(0) = 0. \tag{38}$$

Again, if we put $h = \lambda_0$ in (32) and use (38), we get a product of two determinants of which the 'upper' factor is (37). Thus the existence of Y_0 gives $h = \lambda_0$ as a root of (32). The existence of Y_2 gives (32) itself with $h = \lambda_2$.

6.2. Finally (in the case $m = 3$), if we have a solution

$$Y_3 = a_0+a_1 x+a_2 x^2+x^3,$$

when $\lambda = \lambda_3$, then, for the coefficients of x, x^2, x^3 we have, remembering that now $f_0(3)$, $g_0(3) = 0$,

$$\left.\begin{aligned} 0 &= a_0\{f_1(0)-\lambda_3 g_1(0)\}+a_1\{f_0(1)-\lambda_3 g_0(1)\} \\ 0 &= a_0\{f_2(0)-\lambda_3 g_2(0)\}+a_1\{f_1(1)-\lambda_3 g_1(1)\}+a_2\{f_0(2)-\lambda_3 g_0(2)\} \\ 0 &= \qquad\qquad a_1\{f_2(1)-\lambda_3 g_2(1)\}+a_2\{f_1(2)-\lambda_3 g_1(2)\} \end{aligned}\right\}, \tag{39}$$

and, from the coefficient of x^4,

$$0 = a_2\{f_2(2)-\lambda_3 g_2(2)\}+f_1(3)-\lambda_3 g_1(3). \tag{40}$$

In (39), if we exclude the zero solution a_0, a_1, $a_2 = 0$, we must have the vanishing of the determinant which is (32) with $h = \lambda_3$. Thus (32) has

now the four roots $h = \lambda_0, \lambda_1, \lambda_2, \lambda_3$ and is therefore an identity. In this case the ratios of a_0, a_1, a_2 are determined from (39), while (40) gives the actual value of a_2.

Alternatively $a_2 = 0$ gives, in (40),

$$f_1(3) - \lambda_3 g_1(3) = 0, \tag{41}$$

and, conversely, (41) gives successively $a_2 = 0$, $a_1 = 0$, $a_0 = 0$ since otherwise we should have $\lambda_2 = \lambda_3$, $\lambda_1 = \lambda_3$, or $\lambda_0 = \lambda_3$. Thus to secure the identical vanishing of (32) we need to exclude the possibility $Y_3 \equiv x^3$ and therefore, equivalently, we must exclude (41).

6.3. Generalizing this to any m we can say that the existence of polynomial solutions Y_m ($\not\equiv x^m$) and Y_{m-1} gives to the equation analogous to (32) the roots $h = \lambda_m, \lambda_{m-1}$. If there is a solution Y_n ($n < m-1$), the necessary condition is a *factor* of the extended (32) when $h = \lambda_n$. When all these $m+1$ solutions exist, we get, as above, the $m+1$ roots $h = \lambda_0, \lambda_1, \ldots, \lambda_m$. The exclusion of $Y_m \equiv x^m$ carries with it the condition

$$f_1(m) \neq \lambda_m g_1(m). \tag{42}$$

7. Statement of a theorem

7.1. We can collect all the foregoing into a theorem:

THEOREM 1. *If a system $(F - \lambda G)Y = 0$ of rank two in which $f_0(\delta)$, $g_0(\delta)$ are $\delta(\delta - m)$, m being a positive integer, has polynomial solutions Y_n of all degrees up to and including m, where Y_m itself is not just x^m, then it has polynomial solutions of all degrees, i.e. it is a Sawyer system. Further the differential equation with arbitrary λ has its two series solutions free of logarithms.*

7.2. In connecting the existence of the series S_0, S_m with the existence of the polynomial solutions, it was convenient to use the defining identity because of its symmetry. To determine the form of the f, g necessary for the existence of the polynomial solutions Y_n ($n \leqslant m$) the derived identity is preferable since it enables us to define the f (say) in terms of the g. Thus in

$$[\delta(\delta - m) + x f_1(\delta) + x^2 f_2(\delta)]x^n$$
$$= [\delta(\delta - m) + x g_1(\delta) + x^2 g_2(\delta)](A_0 + \ldots + A_{n-1} x^{n-1} + \lambda_n x^n),$$

when $n < m$, identification of coefficients and elimination of the A gives us, in determinant form, the value of $f_1(n)$.

The case $n = m$ is exceptional. As in § 6.2 above we exclude $Z_n \equiv x^m$

so that $f_1(m) \neq \lambda_m g_1(m)$. The coefficients of $x,\ldots, x^m$ then involve only the g and $A_0,\ldots, A_{m-1}$. These latter can be eliminated to give

$$\begin{vmatrix} g_1(0) & 1-m & 0 & 0 & . & . & . \\ g_2(0) & g_1(1) & 2(2-m) & 0 & . & . & . \\ 0 & g_2(1) & g_1(2) & 3(3-m) & . & . & . \\ . & . & . & . & . & . & . \end{vmatrix} = 0, \tag{43}$$

the determinant being of order m.

8. Relation to the type $\mathscr{A}$

8.1. When $m = 1$, the condition (43) gives $g_1(0) = 0$. Since $Y_0 = 1$ is also a solution, we have further

$$f_1(0) = \lambda_0 g_1(0),$$

so that both $f_1(0)$, $g_1(0)$ vanish. Thus the differential equation of the system has the form

$$[\delta(\delta-1)+x\,\delta h_1(\delta)+x^2 h_2(\delta)]Y = \lambda[\delta(\delta-1)+x\,\delta k_1(\delta)+x^2 k_2(\delta)]Y.$$

This is precisely covered by what in (**1**) § 3.3 I have described as 'type $\mathscr{A}$'. There I had rashly persuaded myself that the general Sawyer system was of this type. This was false. However we see in Theorem 2 its close relationship to forms of type $\mathscr{A}$. I need now to be a little more exact and I shall describe a system as of 'type $\mathscr{A}(m+1)$' when F, G are both of a form

$$\delta\ldots(\delta-m)+\sum_{r=1}^{m} x^r\,\delta\ldots(\delta-m+r)h_r(\delta)+x^{m+1}h_{m+1}(\delta), \tag{44}$$

where the factors $\delta-s$ in the products step down steadily by unity, and $h_1,\ldots, h_{m+1}$ are polynomials with constant coefficients and of degrees indicated by the suffixes.

8.2. I now establish the theorem for equations of rank two:

THEOREM 2. *When $f_0(\delta)$, $g_0(\delta)$ are $\delta(\delta-m)$, then every $F(x,\delta)-\lambda G(x,\delta)$ defining a Sawyer system is an end-factor of some form of type $\mathscr{A}(m+1)$.*

More precisely there is some operator

$$P(x,\delta) \equiv (\delta-1)\ldots(\delta-m+1)+\sum_{r=1}^{m-2} p_r x^r(\delta-1)\ldots(\delta-m+r+1)+p_{m-1}x^{m-1}, \tag{45}$$

in which the p are constants and $p_{m-1} \neq 0$, such that $P(x,\delta)F(x,\delta)$ and $P(x,\delta)G(x,\delta)$ are both of type $\mathscr{A}(m+1)$.

The converse is also true.

Proof. (i) We have

$$P(x,\delta)G(x,\delta) = P(x,\delta)\{\delta(\delta-m)+x\,g_1(\delta)+x^2 g_2(\delta)\}$$
$$= \delta\ldots(\delta-m)+x\,\delta\ldots(\delta-m+2)\{p_1(\delta-m)+g_1(\delta)\}+$$
$$+\sum_{r=1}^{m-1} x^r\,\delta\ldots(\delta-m+r+1)\{p_r(\delta-m)+p_{r-1}g_1(\delta)+p_{r-2}(\delta+1)g_2(\delta)\}+$$
$$+x^m\{p_{m-1}g_1(\delta)+p_{m-2}(\delta+1)g_2(\delta)\}+$$
$$+x^{m+1}p_{m-1}g_2(\delta).$$

For this to be of type $\mathscr{A}(m+1)$ we have to choose the p so that, in the coefficients of $x,\ldots, x^m$, the expressions in $\{\ldots\}$ supply in each case the single factor lacking. In the coefficient of x^r the factor lacking is $\delta-m+r$. Thus we need

$$-rp_r+p_{r-1}g_1(m-r)+(m-r+1)p_{r-2}g_2(m-r) = 0.$$

Write
$$p_r = (m-1)\ldots(m-r)q_r$$

and remove the factor $(m-1)\ldots(m-r+1)$. We have

$$r(r-m)q_r+q_{r-1}g_1(m-r)+q_{r-2}g_2(m-r) = 0. \tag{46}$$

Here r runs from 1 to m, and there are m equations to be satisfied by $q_1,\ldots, q_{m-1}$, necessitating the vanishing of a single determinant. Written with descending r the equations are

$$\left.\begin{aligned} 0 &= q_{m-1}g_1(0)+q_{m-2}g_2(0)\\ 0 &= q_{m-1}(1-m)+q_{m-2}g_1(1)+q_{m-3}g_2(1)\\ &\ldots\ldots\ldots\ldots\ldots\end{aligned}\right\} \tag{47}$$

and the condition is

$$\begin{vmatrix} g_1(0) & g_2(0) & 0 & \ldots\\ 1-m & g_1(1) & g_2(1) & \ldots\\ 0 & 2(2-m) & g_1(2) & \ldots\\ 0 & 0 & 3(3-m) & \ldots\\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = 0, \tag{48}$$

which is exactly (43) transposed. Hence, if G belongs to an $F-\lambda G$ that gives a Sawyer system, so that (43) is satisfied by its components g, then $P(x,\delta)$ can be determined to make PG of type $\mathscr{A}(m+1)$.

Notice that in this determination $p_{m-1}\neq 0$. For $p_{m-1}=0$, i.e. $q_{m-1}=0$, gives in (47) either $g_2(0)=0$ or $p_{m-2}=0$. The first alternative means that $\lambda_0=0$, which has been expressly excluded. But, if $p_{m-1}=0$ and $p_{m-2}=0$, then, again in (47), $g_2(1)=0$ or else $p_{m-3}=0$. Proceeding

in this way we see that $p_{m-1} = 0$ leads to the vanishing of one at least of $\lambda_0,\ldots, \lambda_{m-2}$. But these possibilities have all been excluded. Thus, in such a $P(x,\delta)$

$$p_{m-1} \neq 0. \tag{49}$$

(ii) It remains to show that, with $P(x,\delta)$ so determined, $P(x,\delta)F(x,\delta)$ also is of type $\mathscr{A}(m+1)$, where $F(x,\delta)$ is the other component of the Sawyer form $F-\lambda G$. Operation with P on the defining identity gives

$$P(x,\delta)F(x,\delta)x^n = P(x,\delta)G(x,\delta)Z_n(x). \tag{50}$$

It is characteristic of an operator of type $\mathscr{A}(m+1)$, as is readily seen from (44), that, operating on any polynomial, it leaves behind no power of x below x^{m+1}. Thus, since PG is of that type, no term below x^{m+1} survives in the right-hand member of (50)—and therefore in the left-hand member. Write

$$P(x,\delta)F(x,\delta) \equiv H_0(\delta)+x\,H_1(\delta)+\ldots+x^{m+1}H_{m+1}(\delta).$$

The term of lowest order on the left is $H_0(n)x^n$. Thus we must have $H_0(n) = 0$ when $n < m+1$. This now leaves $H_1(n)x^{n+1}$ as the lowest power on the left. Hence, similarly, $H_1(n) = 0$, when $n < m$, and so on. It follows that, when $r = 0,\ldots, m$, $H_r(\delta)$ contains $\delta\ldots(\delta-m+r)$, so that PF is precisely of type $\mathscr{A}(m+1)$.

This completes the first part of the proof: that every Sawyer form in which f_0, g_0 are $\delta(\delta-m)$ is the end-factor of a form of type $\mathscr{A}(m+1)$.

(iii) The converse is quickly established. Given that $P(F-\lambda G)$ is of type $\mathscr{A}(m+1)$, where expressly $p_{m-1} \neq 0$ as in (49), then, by Theorem A of (1), the differential equation $P(F-\lambda G)Y = 0$ has polynomial solutions Y_n of every degree for appropriate values of λ. Now, if $(F-\lambda_n G)Y_n$ does not itself vanish, it is a polynomial and, in fact, a polynomial solution of $P(x,\delta)y = 0$. But this equation has no polynomial solutions. For, if x^N were the last term in such a solution y_N, $p_{m-1}x^{m+N-1}$ would survive as the last term in $P(x,\delta)y_N$, in the absence of a 'factor of closure' $\delta-N$ in the end-term of $P(x,\delta)$.

Thus every Y_n is a solution of $(F-\lambda_n G)Y = 0$ itself, which is therefore a Sawyer system.

8.3. As was seen in part (i) of the foregoing proof, there are m equations involving $p_1,\ldots, p_{m-1}$ and the values of g_1, g_2. These second-order forms g depend on three constants apiece. Thus, when $m = 6$, the forms g are completely defined in terms of the p; and the determination is unique since the equations are linear in the g. There are identical equations defining f_1, f_2 in terms of the p. Hence, when $m = 6$, F and G are identical, and the system $F-\lambda G$ becomes meaningless. This is true *a fortiori* when

$m > 6$. Thus the only values of consequence are $m = 1, 2, \ldots, 5$, and of these the case $m = 1$ has already been dealt with.

8.4. The technique of determining the form $F - \lambda G$ is straightforward but becomes less simple with increasing m. I do not find the forms significant nor the equations soluble. In the simplest case of $m = 2$ we are led, after a substitution $p_1 x \to x$, to

$$F(x, \delta) \equiv \delta(\delta-2)+x\{(\delta-1)(e\delta+c+2)-(\delta-2)\}+x^2(a\delta^2+b\delta+c), \tag{51}$$

where a, b, c, e are any constants. $G(x, \delta)$ is of similar form with some or all of a, b, c changed.

9. Concluding remarks

9.1. I have said enough about equations of rank two in which g_0 is $\delta(\delta-m)$. The case in which the equation is still of rank two but g_0 is $(\delta-m_0)(\delta-m_0-m)$, where m_0, m are both positive integers, may be briefly dismissed.

For in this case the first series solution of $G(x, \delta)y$ begins with x^{m_0}, and so Z_n $(n \geqslant m_0)$ is divisible by x^{m_0}. Hence, working back, we have every Y_n $(n \geqslant m_0)$ also divisible by x^{m_0}: say $Y_{n+m_0} = x^{m_0} Y'_n$. Thus

$$[F(x, \delta+m_0)-\lambda\, G(x, \delta+m_0)]Y'$$

has polynomial solutions Y'_n of every degree. Here G begins with $\delta(\delta-m)$, and the theory is that given above. Thus the original $F(x, \delta)-\lambda\, G(x, \delta)$ is at least an incomplete Sawyer system of order m_0. To complete it we must impose on it polynomial solutions of degrees $n < m_0$: these are just $Y_n = x^n$. Such further conditions rapidly use up the free constants available in F, G, so that m_0 is narrowly restricted.

9.2. The other case remaining is that in which F, G are of rank exceeding two, while G (and similarly F) has two series solutions. The analysis of § 7 extends readily, I think, to equations of any rank, and we should find that a form $F-\lambda G$ of rank p in which f_0, g_0 are $\delta(\delta-m)$ must be an end-factor of some form of type $\mathscr{A}(m+1)$. It looks therefore as if the problem of determining Sawyer systems of given order is that of determining forms of type $\mathscr{A}$ having end-forms of the required order.

I cannot extend the arguments of § 3 to show that these systems in which G has two solutions are reducible to forms of rank two. I do not think that such a reduction, even if it were possible, would be helpful in view of the increase in order.

9.3. I can answer here a question put to me in correspondence by Professor Burchnall: the Sawyer property of possessing polynomial

solutions is evidently conserved under a substitution $x \to x+a$, where a is constant, but does not this play havoc with any notion of rank? I think that the answer is this.

The type $\mathscr{A}(p)$ has a general form

$$\sum_{r=0}^{p} x^r h_r(\delta)\,\delta(\delta-1)\ldots(\delta-p+r+1),$$

and, as was said in (1), this can be written

$$x^p \sum_{r=0}^{p} h_r(\delta+p-r)D^{m-r}.$$

We can expand each $h_r(\delta+p-r)$ as a polynomial

$$a_0+a_1 xD+a_2 x^2D^2+\ldots.$$

Removal of the common factor x^p and rearrangement gives us a form

$$\sum_{r=0} x^r D^r k_r(D), \tag{52}$$

which is clearly invariant under the substitution $x \to x+a$.

Further, the discussions of § 7 have indicated that in useful applications the rank and order of any $\mathscr{A}(p)$ are the same, namely p. This will be also the order of (52) and its degree in x.

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SOME ASYMPTOTIC FORMULAE IN THE THEORY OF PARTITIONS (II)

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1. Introduction

LET $P(n,k)$ denote the number of partitions of n into at most k integer parts, or what is the same, into parts not exceeding k. In an earlier paper† I have determined the asymptotic behaviour of $P(n,k)$ when k is not too large, viz. $0{\cdot}135k^2 \leqslant n$. In the present paper I shall obtain an asymptotic formula which is valid for arbitrarily large values of k. The main result to be proved is

THEOREM 1. *Let n/k^2 be bounded, $n \leqslant c_1 k^2$, and let β, v be determined from*

$$v\beta = k, \qquad \beta^2 \int_0^v \frac{t}{e^t-1}\,dt+\tfrac{1}{2}\beta\left(\frac{v}{e^v-1}-1\right)+\frac{1}{12}\left(\frac{1}{2}+\frac{1}{e^v-1}-\frac{ve^v}{(e^v-1)^2}\right) = n. \tag{1}$$

Then, uniformly in n and k,

$$P(n,k) = \frac{1}{2\pi}B_0^{-\frac{1}{2}}\beta^{-2}\exp\left(2\beta\int_0^v \frac{t}{e^t-1}\,dt-(v\beta+\tfrac{1}{2})\log(1-e^{-v})+\right.$$

$$\left.+\frac{1}{2}\left(\frac{v}{e^v-1}-1\right)\right)[1+B_1(v)\beta^{-1}+\ldots+B_{m-1}(v)\beta^{-m+1}+O(\beta^{-m})] \tag{2}$$

for any given $m > 0$, where

$$B_0 = \int_0^v \frac{t^2e^t}{(e^t-1)^2}\,dt = 2\int_0^v \frac{t}{e^t-1}\,dt-\frac{v^2}{e^v-1} \tag{3}$$

and $B_\mu(v) = O(1)$.

Remarks:

(i) Throughout the paper, $c_1, c_2, \ldots, C_1, C_2, \ldots$ denote positive constants, independent of k and n but possibly depending on some other previously fixed constants such as the bound for n/k^2, or m in the expansion (2). The same remark applies to constants appearing in the O-notations, e.g. $B_\mu(v) = O(1)$ means $|B_\mu(v)| \leqslant C_\mu$. The letter c without any suffix is reserved for the constant $c = 6^{\frac{1}{2}}/\pi$.

† *Quart. J. of Math.* (Oxford) (2) 2 (1951), 85–108, subsequently quoted as PI.

(ii) $v \geqslant c_2$. For, equation (1) implies

$$v^{-2}\int_0^v \frac{t}{e^t-1}\,dt = n/k^2+\tfrac{1}{2}k^{-1}\left(\frac{1}{v}-\frac{1}{e^v-1}\right)-\tfrac{1}{12}k^{-2}\left(\frac{1}{2}+\frac{1}{e^v-1}-\frac{ve^v}{(e^v-1)^2}\right)$$
$$\leqslant c_1+(4kv)^{-1} \qquad (4)$$

since $$n \leqslant c_1 k^2, \qquad \frac{1}{v}-\frac{1}{e^v-1} \leqslant \frac{1}{2}$$

and the last term is negative for $v > 0$. But the left-hand side of (4) is certainly greater than $1/2v$ when $0 < v < 1$. Hence (4) implies

$$1 < 2c_1 v+(2k)^{-1} < 2c_1 v+\tfrac{1}{2}, \qquad v > (4c_1)^{-1}.$$

(iii) $c_3 n^{\frac{1}{2}} \leqslant \beta \leqslant c_4 n^{\frac{1}{2}}$. This follows from the fact that the coefficients of β^2, β, and β^0 on the left-hand side of (1) are bounded, i.e. $O(1)$ for $v \geqslant 0$, and

$$c_5 \leqslant \int_0^v \frac{t}{e^t-1}\,dt \leqslant c^{-2}, \text{ by remark (ii).}$$

This shows that the expansion (2) proceeds essentially by powers of $n^{-\frac{1}{2}}$.

(iv) It would be possible to determine the functions $B_\mu(v)$ if desired, but it seems difficult to get a general explicit formula. For $B_1(v)$ one obtains

$$B_1(v) = -[\tfrac{1}{12}ve^v(e^v-1)^{-2}+B_0^{-1}\{\tfrac{1}{8}+\tfrac{1}{4}v^2e^v(e^v-1)^{-2}\}+$$
$$+B_0^{-2}\{\tfrac{1}{8}v^4(e^v+e^{2v})(e^v-1)^{-3}-\tfrac{3}{4}v^3e^v(e^v-1)^{-2}\}+$$
$$+\tfrac{5}{24}B_0^{-3}v^6e^{2v}(e^v-1)^{-4}]. \qquad (5)$$

The expression on the right is bounded since $B_0 \geqslant c_6$ for $v \geqslant c_2$ by (3).

(v) Let $v = c^{-1}\lambda n^{\frac{1}{2}}$ $(\lambda \geqslant \delta)$, where δ is a fixed positive number. Then

$$\int_0^v \frac{t}{e^t-1}\,dt = c^{-2}-\int_v^\infty te^{-t}[1+O\{\exp(-c^{-1}\delta n^{\frac{1}{2}})\}]\,dt$$
$$= c^{-2}+O\{n^{\frac{1}{2}}\exp(-c^{-1}\delta n^{\frac{1}{2}})\},$$

and one obtains from (1), (3), and (5)

$$\beta = c(n-\tfrac{1}{24})^{\frac{1}{2}}+\tfrac{1}{4}c^2+\tfrac{1}{32}c^3n^{-\frac{1}{2}}+O(n^{-1}),$$
$$\beta^{-1} = c^{-1}(n-\tfrac{1}{24})^{-\frac{1}{2}}\{1-\tfrac{1}{4}cn^{-\frac{1}{2}}+O(n^{-1})\},$$
$$B_0^{-\frac{1}{2}} = 2^{-\frac{1}{2}}c+O\{n\exp(-c^{-1}\delta n^{\frac{1}{2}})\},$$
$$B_1 = -\tfrac{1}{16}c^2+O\{n^2\exp(-c^{-1}\delta n^{\frac{1}{2}})\}.$$

Hence

$$P(n,k) = \{2^{\frac{3}{2}}\pi c(n-\tfrac{1}{24})\}^{-1}\exp\{2c^{-1}(n-\tfrac{1}{24})^{\frac{1}{2}}\}\{1-\tfrac{1}{2}cn^{-\frac{1}{2}}+O(n^{-1})\}$$

for $k = v\beta = \lambda n+\frac{1}{4}c\lambda n^{\frac{1}{2}}+O(1) \geqslant 2\delta n$. This is the well-known asymptotic term of the Hardy–Ramanujan formula.

The proof of Theorem 1 will be given in the next section. It is entirely independent of the theory of elliptic modular functions and also of Sylvester's theory of 'waves' which formed the core of the proof of my earlier formula in PI. In fact, it depends only on Cauchy's theorem and Euler's summation formula and is essentially an application of the so-called 'method of steepest descent'.

If n/k^2 is not bounded, i.e. $k = o(n^{\frac{1}{2}})$, then formula (2) must be slightly modified. The asymptotic term is still correct, provided that $k \to \infty$, but the error terms have a different form. Formula (29) of § 3 shows that the first error term is $O(k^{-1})$†.

In § 4 I shall investigate $p(n, k)$, the number of partitions of n into exactly k positive integer parts, and $q(n, k)$, the number of partitions of n into k unequal parts. The following is the main result:

THEOREM 2. *If n is sufficiently large, then there exists a number $k_1 = k_1(n)$ such that $p(n, k) < p(n, k+1)$ for $k < k_1$, $p(n, k) > p(n, k+1)$ for $k > k_1$. The value of k_1 is*

$$k_1 = cn^{\frac{1}{2}}L + c^2(\tfrac{3}{2} + \tfrac{3}{2}L - \tfrac{1}{4}L^2) - \tfrac{1}{2} + O(n^{-\frac{1}{2}}\log^4 n), \tag{6}$$

where $L = \log(cn^{\frac{1}{2}})$, $c = 6^{\frac{1}{2}}/\pi = 0{\cdot}7796968\ldots$, $c^2 = 0{\cdot}6079271\ldots$.

THEOREM 3. *If n is sufficiently large, then there exists a number $k_0 = k_0(n)$ such that $q(n, k) < q(n, k+1)$ for $k < k_0$, $q(n, k) > q(n, k+1)$ for $k > k_0$. The value of k_0 is*

$$k_0 = (2^{\frac{1}{2}}\log 2)cn^{\frac{1}{2}} + 2b(\log 2)^{-1} - \frac{\frac{1}{2}b}{1-2b} - 1 + O(n^{-\frac{1}{2}}), \tag{7}$$

where $b = c^2(\log 2)^2$.

Numerically, $k_0 = 0{\cdot}7643041n^{\frac{1}{2}} - 0{\cdot}5084280 + O(n^{-\frac{1}{2}})$.

Theorem 2 proves a conjecture of Auluck, Chowla, and Gupta‡, and improves a formula of Erdös§ who showed that

$$k_1 = cn^{\frac{1}{2}}\log(cn^{\frac{1}{2}}) + o(n^{\frac{1}{2}}).$$

It also disproves a conjecture I made in PI, 94.

Theorem 3 was essentially proved in PI. My earlier k_0 was greater by $\frac{1}{2}$ than the present value, owing to a different interpretation of the maxima.

† The formula in PI had a different form and proceeded according to powers of k^{-1}.

‡ *J. Indian Math. Soc.* 6 (1942), 105–12.

§ *Bull. American Math. Soc.* 52 (1946), 185–8. Erdös uses the notation $c = \pi(\frac{2}{3})^{\frac{1}{2}}$.

2. Proof of Theorem 1

Let
$$F(w) = \prod_{\nu=1}^{k} (1-w^\nu)^{-1} = \sum_{n=0}^{\infty} P(n,k) w^n. \tag{8}$$

By Cauchy's theorem

$$P(n,k) = (2\pi i)^{-1} \int F(w) w^{-n-1}\, dw \tag{9}$$

taken along a circle with centre at O and radius $\rho < 1$. I choose ρ to be the real positive root of the equation $w F'/F(w) = n$, i.e.

$$\sum_{\nu=1}^{k} \frac{\nu\rho^\nu}{1-\rho^\nu} = \sum_{\nu=1}^{k} \frac{\nu}{e^{\nu\alpha}-1} = n, \qquad \alpha = -\log\rho. \tag{10}$$

The essential point is that with this choice of ρ we shall be able to evaluate the integral (9) quite accurately in the neighbourhood of the 'saddle point' $w = \rho$ and then show that the more remote parts of the path do not contribute substantially to the value of the integral. This can be done quite easily, without having recourse to Farey dissections or similar devices. Thus the influence of the secondary singularities is automatically eliminated.

Throughout this section the symbol $A(u)$ denotes some unspecified continuous function of u, bounded for $u \geqslant 0$, which, if desired, can be determined explicitly. In the following I shall frequently use Euler's summation formula†

$$f(1)+\ldots+f(k) = \int_0^k f(x)\, dx + \tfrac{1}{2}\{f(k)-f(0)\} +$$
$$+ \sum_{\nu=1}^{m} \frac{1}{(\nu+1)!} B_{\nu+1}\{f^{(\nu)}(k)-f^{(\nu)}(0)\} + \int_0^k P_{m+1}(x) f^{(m+1)}(x)\, dx,$$

where $B_2 = \frac{1}{6}$, $B_3 = 0,\ldots$, generally B_ν is the νth Bernoulli number, and $P_m(t)$ is the mth Bernoulli polynomial.

Take
$$f(x) = x(e^{\alpha x}-1)^{-1} = \frac{1}{\alpha} t(e^t-1)^{-1} = \frac{1}{\alpha}\phi(t),$$

where $t = \alpha x$, so that

$$f^{(\nu)}(x) = \alpha^{\nu-1}\phi^{(\nu)}(t) \quad \text{and} \quad f^{(\nu)}(0) = \alpha^{\nu-1} B_\nu.$$

† See K. Knopp, *Infinite Series* (London 1928), 526.

Clearly $\phi^{(\nu)}(t)$ is bounded for $t \geqslant 0$ and tends to zero monotonically for $t \geqslant t_\nu$. Hence, for a fixed m,

$$\int_0^\infty |\phi^{(m+1)}(t)|\, dt = O(1)$$

and

$$\int_0^k |P_{m+1}(x) f^{(m+1)}(x)|\, dx = \alpha^{m-1} \int_0^{\alpha k} \left| P_{m+1}\left(\frac{1}{\alpha} t\right) \phi^{(m+1)}(t) \right| dt = O(\alpha^{m-1}).$$

On writing $u = \alpha k$, Euler's formula and equation (10) give

$$n = \sum_{\nu=1}^{k} \nu (e^{\nu\alpha} - 1)^{-1} = \alpha^{-2} \int_0^u \frac{t}{e^t - 1}\, dt + \tfrac{1}{2}\alpha^{-1}\left(\frac{u}{e^u - 1} - 1\right) +$$

$$+ \frac{1}{12}\left(\frac{1}{e^u - 1} - \frac{u e^u}{(e^u - 1)^2} + \frac{1}{2}\right) + \alpha^2 A(u) + \ldots + \alpha^{m-2} A(u) + O(\alpha^{m-1}). \quad (11)$$

Generally, for $0 \leqslant s < r$, one obtains

$$\sum_{\nu=1}^{k} \nu^r e^{s\alpha\nu} (e^{\alpha\nu} - 1)^{-r} = \alpha^{-r-1} \int_0^u t^r e^{st} (e^t - 1)^{-r}\, dt +$$

$$+ \alpha^{-r} A(u) + \ldots + \alpha^{-r+m-1} A(u) + O(\alpha^{-r+m}). \quad (12)$$

Comparing (1) and (11) we find

$$\left.\begin{aligned} \alpha &= \beta^{-1}\{1 + A(v)\beta^{-4} + \ldots + A(v)\beta^{-m+1} + O(\beta^{-m})\} \\ u &= v\{1 + A(v)\beta^{-4} + \ldots + A(v)\beta^{-m+1} + O(\beta^{-m})\} \end{aligned}\right\}. \quad (13)$$

The second relation is a consequence of the first one since $u = k\alpha = \alpha\beta v$. To prove the first relation, put $\alpha = \beta^{-1}(1+\epsilon)$, $u = v(1+\epsilon)$ into (11) and subtract equation (1). The result is

$$0 = -2\beta^2\epsilon \int_0^v \frac{t}{e^t - 1}\, dt + \beta^2\epsilon \frac{v^2}{e^v - 1} + \beta^2 O(\epsilon^2) + \beta O(\epsilon) + \beta^{-2} A(v)$$

$$= -\beta^2\epsilon \int_0^v t^2 e^t (e^t - 1)^{-2}\, dt + \beta^{-2} A(v) + \beta^2 O(\epsilon^2) + \beta O(\epsilon),$$

which implies $\epsilon = \beta^{-4}A(v)+O(\beta^{-5})$. This proves (13) for $m = 4$. In the general case we use induction on m. Writing

$$\epsilon = \beta^{-4}A(v)+\ldots+\beta^{-m+1}A(v)+\epsilon_1, \qquad \epsilon_1 = O(\beta^{-m})$$

one obtains as before

$$-\beta^2\epsilon_1 \int_0^v t^2e^t(e^t-1)^{-2}\,dt+\beta^{-m+2}A(v)+O(\beta^{-m+1}) = 0,$$

where $A(v)$ can be determined recursively. It follows that ϵ_1 has the form $\beta^{-m}A(v)+O(\beta^{-m-1})$.

From (8) and (9) we get

$$P(n,k) = (2\pi)^{-1}\rho^{-n}F(\rho)\int_{-\pi}^{\pi}\frac{F(\rho e^{i\theta})}{F(\rho)}e^{-ni\theta}\,d\theta$$

$$= (2\pi)^{-1}\exp(n\alpha+\log F(\rho))\int_{-\pi}^{\pi}\exp\left\{-\sum_{\nu=1}^{k}\log\frac{1-\rho^\nu e^{i\nu\theta}}{1-\rho^\nu}-ni\theta\right\}d\theta. \quad (14)$$

The integral is dissected in three parts:

$$\int_{-\pi}^{\pi} = I_1+I_2+I_3, \qquad I_1 = \int_{-\theta_0}^{\theta_0}, \qquad I_2 = \int_{\theta_0}^{\pi}, \qquad I_3 = \int_{-\pi}^{-\theta_0}, \quad (15)$$

where $\theta_0 = n^{-5/7}$.

For the integrand in I_1 we can write

$$\exp\left[-\left\{\sum_{\nu=1}^{k'}+\sum_{\nu=k'+1}^{k}\right\}\log\frac{1-\rho^\nu e^{i\nu\theta}}{1-\rho^\nu}-ni\theta\right], \quad (16)$$

where $k' = \min\{k,[n^{\frac{1}{2}}]\}$.† The second sum in (16) is empty unless $k > n^{\frac{1}{2}}$, when $(k'+1)\alpha > c_7n^{\frac{1}{2}} > 1$ for $n > c_8$ and

$$\left|\sum_{\nu=k'+1}^{k}\right| \leqslant \sum_{\nu=k'+1}^{\infty}\log\{1+2(e^{\alpha\nu}-1)^{-1}\} < 2\sum_{\nu=k'+1}^{\infty}(e^{\alpha\nu}-1)^{-1}$$

$$< 3\sum_{\nu=k'+1}^{\infty}e^{-\alpha\nu} = O(e^{-\alpha(k'+1)})$$

$$= O\{\exp(-c_9n^{\frac{1}{2}})\}. \quad (17)$$

† $[x]$ denotes the greatest integer not exceeding x.

Since $|\nu\theta| \leqslant k'\theta_0 \leqslant n^{-1/21}$, the first sum in (16) can be expanded in powers of $i\nu\theta$,

$$\sum_{\nu=1}^{k'} \log\frac{1-\rho^\nu e^{i\nu\theta}}{1-\rho^\nu}$$

$$= -\sum_{\nu=1}^{k'} \log\Big\{1-(e^{\nu\alpha}-1)^{-1}\sum_{\mu=1}^{m}\frac{1}{\mu!}(i\nu\theta)^\mu + O[(\nu|\theta|)^{m+1}(e^{\nu\alpha}-1)^{-1}]\Big\}$$

$$= i\theta\sum_{\nu=1}^{k'}\frac{\nu}{e^{\nu\alpha}-1} - \tfrac{1}{2}\theta^2\sum_{\nu=1}^{k'}\frac{\nu^2 e^{\nu\alpha}}{(e^{\nu\alpha}-1)^2} + \sum_{\mu=3}^{2m+1}\frac{1}{\mu!}(i\theta)^\mu\sum_{\nu=1}^{k'}\frac{\nu^\mu g_\mu(e^{\nu\alpha})}{(e^{\nu\alpha}-1)^\mu} +$$

$$+ O\Big\{\theta^{2m+2}\sum_{\nu=1}^{\infty}\nu^{2m+2}\sum_{\mu=1}^{2m+2}(e^{\nu\alpha}-1)^{-\mu}\Big\}, \quad (18)$$

where $g_3(t) = t+t^2$, $g_4(t) = t+4t^2+t^3$, and generally $g_\mu(t)$ is a polynomial of degree $\mu-1$. Replacing the upper summation index k' by k, we see that the error is

$$O\Big(\int_{n^{2/3}}^{\infty}\frac{dt}{e^{t\alpha}}\Big) = O\{n^{\frac{1}{3}}\exp(-c_{10}\,n^{\frac{1}{3}})\},$$

and the first sum on the right of (18) becomes $ni\theta$.† The other sums can be replaced by the corresponding integrals, with the help of Euler's formula [equation (12)]. The result is

$$ni\theta - \tfrac{1}{2}\theta^2\alpha^{-3}\int_0^u\frac{t^2e^t}{(e^t-1)^2}\,dt - \theta^2\alpha^{-3}\sum_{\nu=1}^{n-1}A(u)\alpha^\nu +$$

$$+\sum_{\mu=3}^{2m+1}(i\theta)^\mu\alpha^{-\mu-1}\sum_{\nu=0}^{m-1}A(u)\alpha^\nu + O(\theta^2\alpha^{m-3}+\theta^{2m+2}\alpha^{-2m-3}).$$

This and (15), (16), (17) give

$$I_1 = \int_{-\theta_0}^{\theta_0}\exp\Big(-\tfrac{1}{2}\theta^2\alpha^{-3}\int_0^u\frac{t^2e^t}{(e^t-1)^2}\,dt\Big)\times$$

$$\times\Big\{1+\sum_{\mu=1}^{m}\theta^{2\mu}\alpha^{-[8\mu/3]}\sum_{\nu=0}^{m-1}A(u)\alpha^\nu + ih(\theta,\alpha) + O\Big(\sum_{\nu=1}^{m-1}\theta^{2\nu}\alpha^{m-3\nu}\Big)\Big\}\,d\theta,$$

where $h(\theta,\alpha)$ is an odd function of θ. Hence, writing

$$A_0 = \int_0^u\frac{t^2e^t}{(e^t-1)^2}\,dt, \qquad x = A_0^{\frac{1}{2}}(2\alpha^3)^{-\frac{1}{2}}\theta,$$

† This is the decisive step where the saddle-point property of ρ is used.

we get

$$I_1 = (2\alpha^3 A_0^{-1})^{\frac{1}{2}} \int_{-x_0}^{x_0} e^{-x^2}\Big\{1+\sum_{\mu=1}^{n}\Big(\frac{2}{A_0}\Big)^{\mu}\alpha^{[(\mu+2)/3]}x^{2\mu}\sum_{\nu=0}^{m-1}A(u)\alpha^{\nu}+ \\ +O\Big(\alpha^m\sum_{\nu=1}^{m+1}x^{2\nu}\Big)\Big\}\,dx,$$

where $x_0 = A_0^{\frac{1}{2}}(2\alpha^3)^{-\frac{1}{6}}\theta_0 \geqslant c_{11}\,n^{3/4-5/7} = c_{11}\,n^{1/28}$.

Replacing the limits $\pm x_0$ by $\pm\infty$, we see that the error is at most

$$O\Big\{\int_{x_0}^{\infty} x^{2m+2}e^{-x^2}\,dx\Big\} = O\{n\exp(-c_{12}\,n^{1/14})\}.$$

Thus $$I_1 = (2\pi\alpha^3 A_0^{-1})^{\frac{1}{2}}\{1+A_1(u)\alpha+\ldots+A_{m-1}(u)\alpha^{m-1}+O(\alpha^m)\} \qquad (19)$$

for certain functions $A_\mu(u)$ which are bounded for $u \geqslant c_{13}$. In fact, each $A_\mu(u)$ can be expressed as a fraction $a_\mu(u)\{A_0(u)\}^{-3\mu}$, where $a_\mu(u)$ is bounded for $u \geqslant 0$ and $A_0(u) \geqslant c_{14}$ for $u \geqslant c_{13}$. A simple calculation gives, e.g.

$$-A_1(u) = A_0^{-1}\Big(\frac{1}{8}+\frac{1}{4}\frac{u^2e^u}{(e^u-1)^2}\Big)+A_0^{-2}\Big(\frac{1}{8}\frac{u^4(e^u+e^{2u})}{(e^u-1)^3}-\frac{3}{4}\frac{u^3e^u}{(e^u-1)^2}\Big)+ \\ +\frac{5}{24}A_0^{-3}\frac{u^6e^{2u}}{(e^u-1)^4}. \qquad (20)$$

It remains to be shown that I_2 and I_3 in (15) are negligible. For the absolute value $G(\theta)$ of the integrand in I_2 one has, if $n^{-5/7} \leqslant \theta \leqslant n^{-1/2}$ and $k' = \min(k, [\alpha^{-1}], [\pi/4\theta])$,

$$G(\theta) = \Big|\exp\Big(\sum_{\nu=1}^{k}\log\frac{1-\rho^\nu}{1-\rho^\nu e^{i\nu\theta}}\Big)\Big| \\ \leqslant \exp\Big[-\sum_{\nu=1}^{k'}\tfrac{1}{2}\log\{1+2e^{\nu\alpha}(e^{\nu\alpha}-1)^{-2}(1-\cos\nu\theta)\}\Big] \\ \leqslant \exp\Big[-\tfrac{1}{2}\sum_{\nu=1}^{k'}\log\{1+\nu^2\theta^2/(8\nu^2\alpha^2)\}\Big] = \exp[-\tfrac{1}{2}k'\log\{1+\theta^2/(8\alpha^2)\}],$$

$$G(\theta) < \exp(-c_{15}n^{1/14}), \qquad (21)$$

since

$$\tfrac{1}{2}k'\log(1+\theta^2/8\alpha^2) > \tfrac{1}{32}k'\theta^2\alpha^{-2} \geqslant \tfrac{1}{32}[\alpha]^{-3}\theta_0^2 > c_{16}\,n^{3/2-10/7} \\ = c_{16}\,n^{1/14} \quad \text{for } \theta \leqslant 8\alpha$$

and again

$$\tfrac{1}{2}k'\log(1+\theta^2/8\alpha^2) > \tfrac{1}{2}k'\log(\theta/4\alpha) > c_{17}\,n^{\frac{1}{2}} \quad \text{for } 8\alpha < \theta \leqslant n^{-\frac{1}{2}}.$$

If $n^{-\frac{1}{2}} < \theta \leqslant \pi$, then for every positive integer $m \leqslant m_0 = k'\theta/2\pi$, where $k' = \min(k, [\alpha^{-1}])$, there is at least one $\nu \leqslant k'$ with

$$(2m-\tfrac{3}{2})\pi \leqslant \nu\theta \leqslant (2m-\tfrac{1}{2})\pi,$$

and, for this ν,

$$\log|(1-\rho^\nu)(1-\rho^\nu e^{i\nu\theta})^{-1}| \leqslant \tfrac{1}{2}\log(1-\rho^\nu)^2(1+\rho^{2\nu})^{-1}$$
$$< \tfrac{1}{2}\log(e^{2\pi\alpha m}-1)^2(e^{4\pi\alpha m}+1)^{-1}.$$

Hence

$$\log G(\theta) \leqslant \tfrac{1}{2}\sum_{m\leqslant m_0}\log(e^{2\pi\alpha m}-1)^2(e^{4\pi\alpha m}+1)^{-1}$$
$$\leqslant \tfrac{1}{2}\int_1^{m_0}\log(e^{2\pi\alpha x}-1)^2(e^{4\pi\alpha x}+1)^{-1}\,dx$$
$$= (4\pi^\alpha)^{-1}\int_{2\pi\alpha}^{2\pi\alpha m_0}\log\frac{(e^t-1)^2}{e^{2t}+1}\,dt < -c_{18}\,n^{\frac{1}{2}}$$

since $2\pi\alpha m_0 = \alpha k'\theta \geqslant c_{19}\,n^{-\frac{1}{2}}$ and $2\pi\alpha \leqslant c_{20}\,n^{-\frac{1}{2}}$.

It follows that

$$G(\theta) < \exp(-c_{18}\,n^{\frac{1}{2}}) \quad \text{for } n^{-\frac{1}{2}} < \theta \leqslant \pi. \tag{22}$$

Now (21) and (22) give

$$G(\theta) < \exp(-c_{21}\,n^{1/14}) \quad \text{for } \theta_0 \leqslant \theta \leqslant \pi$$

and

$$|I_2| = |I_3| < 2\pi\exp(-c_{21}\,n^{1/14}). \tag{23}$$

Summarizing, (8), (10), (14), (15), (19), (23) give

$$P(n,k) = (2\pi A_0)^{-\frac{1}{2}}\alpha^{\frac{1}{2}}\exp\Big[\sum_{\nu=1}^{k}\Big\{\frac{\nu\alpha}{e^{\nu\alpha}-1}-\log(1-e^{-\nu\alpha})\Big\}\Big]\times$$
$$\times\{1+A_1(u)\alpha+\ldots+A_{m-1}(u)\alpha^{m-1}+O(\alpha^m)\}, \tag{24}$$

where $A_1(u)$ is given by (20).

To evaluate the expression in the exponential we once more use Euler's formula

$$\sum_{\nu=1}^{k}\log\frac{\nu}{1-e^{-\nu\alpha}} = \int_0^k\log\frac{x}{1-e^{-\alpha x}}\,dx+\tfrac{1}{2}\{\log k-\log(1-e^{-u})+\log\alpha\}+$$
$$+\tfrac{1}{12}\alpha\Big(\frac{1}{u}-\frac{1}{e^u-1}-\frac{1}{2}\Big)+\sum_{\mu=3}^{m-1}A(u)\alpha^\mu+O(\alpha^m)$$
$$= (k+\tfrac{1}{2})\log k-k+\alpha^{-1}\int_0^u\frac{t}{e^t-1}\,dt-(k+\tfrac{1}{2})\log(1-e^{-u})+$$
$$+\tfrac{1}{2}\log\alpha+\tfrac{1}{12}\alpha\Big(\frac{1}{u}-\frac{1}{e^u-1}-\frac{1}{2}\Big)+\sum_{\mu=3}^{m-1}A(u)\alpha^\mu+O(\alpha^m).$$

This with Stirling's formula† gives

$$-\sum_{\nu=1}^{k}\log(1-e^{-\nu\alpha})=\sum_{\nu=1}^{k}\log\frac{\nu}{1-e^{-\nu\alpha}}-\log k!$$

$$=\frac{1}{\alpha}\int_0^u\frac{t}{e^t-1}\,dt-\left(\frac{u}{\alpha}+\frac{1}{2}\right)\log(1-e^{-u})+\tfrac{1}{2}\log\frac{\alpha}{2\pi}-$$

$$-\tfrac{1}{12}\alpha\left(\frac{1}{2}+\frac{1}{e^u-1}\right)+\sum_{\mu=3}^{m-1}A(u)\alpha^\mu+O(\alpha^m).$$

Formula (2) with (4) is obtained from this, (11), and (24), by using (13) and (20).

3. The case $k=O(n^{\frac{1}{2}})$

We now drop the assumption that n/k^2 is bounded and suppose that $k\leqslant c_{22}n^{\frac{1}{2}}$ but $k\to\infty$. Equation (11) gives $u\leqslant c_{23}$ and

$$n=k^2u^{-2}\int_0^u\frac{t}{e^t-1}\,dt+\tfrac{1}{2}k\left(\frac{1}{e^u-1}-\frac{1}{u}\right)+$$

$$+\frac{1}{12}\left(\frac{1}{e^u-1}+\frac{1}{2}-\frac{ue^u}{(e^u-1)^2}\right)+O(u^2k^{-2})$$

$$=k^2u^{-1}B_1(u)+kB_2(u)+\tfrac{1}{12}uB_3(u)+O(u^2k^{-2}),\qquad(25)$$

where B_1, B_2, B_3 are bounded for $u\geqslant 0$, and $B_1(u)\geqslant c_{24}$ for $0\leqslant u\leqslant c_{23}$. Therefore

$$u=O(k^2n^{-1}),\quad u^{-1}=O(nk^{-2}),\quad \alpha=u/k=O(k/n),\quad \alpha^{-1}=O(n/k),\qquad(26)$$

and the error term in (25) is $O(k^2n^{-2})$.

If v, β are determined from

$$v=\beta k,\qquad n=k^2v^{-2}B_1(v)+kB_2(v)+\tfrac{1}{12}vB_3(v),\qquad(27)$$

then

$$u=v\{1+O(k^2n^{-3})\}=v\{1+O(n^{-2})\},\qquad \alpha=\beta^{-1}\{1+O(n^{-2})\}.\qquad(28)$$

† See K. Knopp, *Infinite Series*, 531.

To carry out the dissection (15) we take $\theta_0 = n^{-1}k^{4/7}$. Then $k\theta_0 = O(k^{-3/7})$ and (16) becomes

$$\exp\Big\{-\sum_{\nu=1}^{k}\log\frac{1-\rho^\nu e^{i\nu\theta}}{1-\rho^\nu}-ni\theta\Big\}$$

$$=\exp\Big[-\sum_{\nu=1}^{k}\log\Big\{1-\frac{1}{e^{\nu\alpha}-1}(i\nu\theta-\tfrac{1}{2}\nu^2\theta^2-\tfrac{1}{6}i\nu^3\theta^3)+O\Big(\frac{\nu^4\theta^4}{e^{\nu\alpha}-1}\Big)\Big\}-ni\theta\Big]$$

$$=\exp\Big[-\tfrac{1}{2}\theta^2\sum_{\nu=1}^{k}\frac{\nu^2e^{\nu\alpha}}{(e^{\nu\alpha}-1)^2}-\tfrac{1}{6}i\theta^3\sum_{\nu=1}^{k}\frac{\nu^3(e^{2\nu\alpha}+e^{\nu\alpha})}{(e^{\nu\alpha}-1)^3}+$$

$$+O\Big\{\theta^4\sum_{\nu=1}^{\infty}\nu^4\sum_{\mu=1}^{4}(e^{\nu\alpha}-1)^{-\mu}\Big\}\Big]$$

$$=\exp\Big\{-\tfrac{1}{2}\theta^2\alpha^{-3}\int_0^u\frac{t^2e^t}{(e^t-1)^2}\,dt\Big\}\Big\{1-\tfrac{1}{6}i\theta^3\alpha^{-4}\int_0^u\frac{t^3(e^{2t}+e^t)}{(e^t-1)^3}\,dt+$$

$$+O(n\theta^2+n^4k^{-3}\theta^4+n^6k^{-4}\theta^6)\Big\}.$$

This gives, since $x_0 = (\frac{1}{2}A_0)^{\frac{1}{2}}\alpha^{-\frac{1}{2}}\theta_0 \geqslant c_{24}\,k^{1/14}$,

$$I_1 = (2\pi\alpha^3A_0^{-1})^{\frac{1}{2}}\{1+O(k^{-1})\}.$$

To show that I_2 is small, we can use the same estimates as for (21) and (22) except that the intervals for θ are now $\theta_0 \leqslant \theta \leqslant k^{-\frac{1}{2}}$ and $k^{-\frac{1}{2}} < \theta \leqslant \pi$ respectively. Hence

$$P(n,k) = (2\pi A_0)^{-\frac{1}{2}}\alpha^{\frac{1}{2}}\exp\Big\{\sum_{\nu=1}^{k}\Big(\frac{\nu\alpha}{e^{\nu\alpha}-1}\Big)-\log(1-e^{-\nu\alpha})\Big\}\{1+O(k^{-1})\}$$

$$=\frac{v^2}{2\pi k^2}B_0^{-\frac{1}{2}}\exp\Big\{\frac{2k}{v}\int_0^v\frac{t}{e^t-1}\,dt+\frac{1}{2}\Big(\frac{v}{e^v-1}-1\Big)-$$

$$-(k+\tfrac{1}{2})\log(1-e^{-v})\Big\}\{1+O(k^{-1})\}, \quad (29)$$

where $B_0 = \int_0^v\frac{t^2e^t}{(e^t-1)^2}\,dt$ and v is determined from (27). This is valid uniformly in k provided that $k\to\infty$ and $k \leqslant c_{22}\,n^{\frac{1}{2}}$.

4. The functions $p(n, k)$ and $q(n, k)$

Formulae (2) and (29) respectively can be used as asymptotic expressions for $p(n, k)$ and $q(n, k)$, in virtue of the elementary relations

$$p(n,k) = P(n-k,k), \qquad q(n,k) = P\left(n-\binom{k+1}{2}, k\right). \tag{30}$$

All we have to do is to replace the right-hand side of (1) by $n-k$ and $n-\binom{k+1}{2}$ respectively.

Consider first $p(n, k)$ and suppose that n/k^2 is bounded. Without loss of generality we may assume that $k < \frac{1}{2}n$ since $p(n, k) = P(n-k, k)$ for $k \geqslant \frac{1}{2}n$ by (30). (Hence $p(n, k)$ is steadily decreasing for $k \geqslant \frac{1}{2}n$ when n is fixed.) The last assumption implies that remark (iii) of the introduction is valid.

To obtain the position of the maximum of $p(n, k)$ when n is fixed, it is convenient to go back to the relations

$$\sum_{\nu=1}^{k} \frac{\nu}{e^{\nu\alpha}-1} = n-k, \tag{31}$$

$$p(n,k) = (2\pi A_0)^{-\frac{1}{2}}\alpha^{\frac{1}{2}}\exp\left\{\sum_{\nu=1}^{k}\frac{\nu\alpha}{e^{\nu\alpha}-1} - \log(1-e^{-\nu\alpha})\right\}\times$$

$$\times\{1+A_1(u)\alpha+A_2(u)\alpha^2+O(\alpha^3)\}, \tag{32}$$

which follow from the equations (10) and (24).

Let $$\sum_{\nu=1}^{k+1} \frac{\nu}{e^{\nu(\alpha+\Delta\alpha)}-1} = n-k-1.$$

By (31) and Taylor's theorem,

$$\sum_{\nu=1}^{k} \frac{\nu}{e^{\nu\alpha}-1} - \sum_{\nu=1}^{k+1} \frac{\nu}{e^{\nu(\alpha+\Delta\alpha)}-1} = \sum_{\nu=1}^{k} \frac{\nu^2 e^{\nu\alpha}}{(e^{\nu\alpha}-1)^2}\Delta\alpha - \frac{k+1}{e^{(k+1)\alpha}-1} +$$

$$+O\{u^2e^u\Delta\alpha+\alpha^{-4}(\Delta\alpha)^2\} = 1, \tag{33}$$

$$A_0\alpha^{-3}\Delta\alpha-\alpha^{-1}\frac{u}{e^u-1} = 1+O(ue^{-u}+\alpha^{-2}\Delta\alpha),$$

$$\Delta\alpha = A_0^{-1}\alpha^2\left(\frac{u}{e^u-1}+\alpha\right)+O(ue^{-u}\alpha^3+\alpha^4). \tag{34}$$

Also $u = \alpha k$, $u+\Delta u = (\alpha+\Delta\alpha)(k+1)$,

$$\Delta u = k\Delta\alpha+\alpha+\Delta\alpha = \frac{u}{\alpha}\Delta\alpha+\alpha+\Delta\alpha. \tag{35}$$

Furthermore,

$$\begin{aligned}
&\Delta\sum_{\nu=1}^{k}\left\{\frac{\nu\alpha}{e^{\nu\alpha}-1}-\log(1-e^{-\nu\alpha})\right\}\\
&\quad = -\alpha-\log(1-e^{-u})-\frac{\alpha}{e^{u}-1}-\tfrac{1}{2}A_0\alpha^{-3}(\Delta\alpha)^2+O(\Delta\alpha)\\
&\quad = -\log(1-e^{-u})-\alpha\left\{1+\frac{1}{e^{u}-1}+\tfrac{1}{2}A_0^{-1}\frac{u^2}{(e^{u}-1)^2}\right\}+O(\Delta\alpha) \qquad (36)
\end{aligned}$$

since

$$\begin{aligned}
&\Delta\left\{\sum_{\nu=1}^{k}\frac{\nu\alpha}{e^{\nu\alpha}-1}\right\}\\
&\quad = -\alpha+\sum_{\nu=1}^{k+1}\frac{\nu}{e^{\nu(\alpha+\Delta\alpha)}-1}\Delta\alpha\\
&\quad = -\alpha+\left(\sum_{\nu=1}^{k}\frac{\nu}{e^{\nu\alpha}-1}\right)\Delta\alpha-\sum_{\nu=1}^{k}\frac{\nu^2e^{\nu\alpha}}{(e^{\nu\alpha}-1)^2}(\Delta\alpha)^2+\\
&\qquad\qquad +\frac{k+1}{e^{(k+1)(\alpha+\Delta\alpha)}-1}\Delta\alpha+O\{\alpha^{-4}(\Delta\alpha)^3\}\\
&\quad = -\alpha+\left(\sum_{\nu=1}^{k}\frac{\nu}{e^{\nu\alpha}-1}\right)\Delta\alpha-\frac{k}{e^{u}-1}\Delta\alpha-A_0\alpha^{-3}(\Delta\alpha)^2+O(\Delta\alpha),
\end{aligned}$$

by (33) and (34), and

$$\begin{aligned}
&\Delta\left\{-\sum_{\nu=1}^{k}\log(1-e^{-\nu\alpha})\right\}\\
&\quad = -\log(1-e^{-(k+1)(\alpha+\Delta\alpha)})-\left(\sum_{\nu=1}^{k}\frac{\nu}{e^{\nu\alpha}-1}\right)\Delta\alpha+\\
&\qquad\qquad +\frac{1}{2}\sum_{\nu=1}^{k}\frac{\nu^2e^{\nu\alpha}}{(e^{\nu\alpha}-1)^2}(\Delta\alpha)^2+O\{\alpha^{-4}(\Delta\alpha)^3\}\\
&\quad = -\left(\sum_{\nu=1}^{k}\frac{\nu}{e^{\nu\alpha}-1}\right)\Delta\alpha-\log(1-e^{-u})-\frac{\alpha}{e^{u}-1}-\frac{k}{e^{u}-1}\Delta\alpha+\\
&\qquad\qquad +\tfrac{1}{2}A_0\alpha^{-3}(\Delta\alpha)^2+O(\Delta\alpha).
\end{aligned}$$

Finally

$$\tfrac{3}{2}\{\log(\alpha+\Delta\alpha)-\log\alpha\} = \tfrac{3}{2}\alpha^{-1}\Delta\alpha+O\{\alpha^{-2}(\Delta\alpha)^2\}$$

$$= \tfrac{3}{2}A_0^{-1}\alpha\frac{u}{e^u-1}+\tfrac{3}{2}A_0^{-1}\alpha^2+O(\Delta\alpha), \qquad (37)$$

$$-\tfrac{1}{2}\{\log A_0(u+\Delta u)-\log A_0(u)\}$$

$$= -\tfrac{1}{2}A_0^{-1}\frac{u^2e^u}{(e^u-1)^2}\left(\frac{u}{\alpha}\Delta\alpha+\alpha\right)+O\{(\Delta u)^2\}$$

$$= -\frac{\alpha}{2}\left\{A_0^{-1}\frac{u^2e^u}{(e^u-1)^2}+A_0^{-2}\frac{u^4e^u}{(e^u-1)^3}\right\}+O(u^2\Delta\alpha), \quad \text{by (35)}, \qquad (38)$$

and
$$\Delta\{\alpha A_1(u)+\alpha^2A_2(u)\} = O(u^2\Delta\alpha) \qquad (39)$$

by (20), (34), and (35). Hence

$$\Delta\log p(n,k) = \log p(n,k+1)-\log p(n,k)$$

$$= -\log(1-e^{-u})-\alpha\left[1+\frac{1}{e^u-1}+\tfrac{1}{2}A_0^{-1}\frac{u^2(e^u+1)}{(e^u-1)^2}-\tfrac{3}{2}A_0^{-1}\frac{u}{e^u-1}+\right.$$

$$\left.+\tfrac{1}{2}A_0^{-2}\frac{u^4e^u}{(e^u-1)^3}\right]+\tfrac{3}{2}A_0^{-1}\alpha^2+O(u^2\Delta\alpha) \qquad (40)$$

by (32), (36), (37), (38), and (39).

Put $u = \frac{1}{2}\log n+\lambda$, where λ is some function of n which either tends to $+\infty$ or $-\infty$ or remains bounded.

Since $c_{25}n^{-\frac{1}{2}} \leqslant \alpha \leqslant c_{26}n^{-\frac{1}{2}}$, the right-hand side of (40) is positive for sufficiently large n if $e^u = o(n^{\frac{1}{2}})$, i.e. if $\lambda \to -\infty$, and is negative if $e^{-u} = o(n^{-\frac{1}{2}})$, i.e. if $\lambda \to +\infty$. Hence we may assume that λ is bounded. Using

$$\alpha^{-2}\int_0^u \frac{t}{e^t-1}\,dt+\tfrac{1}{2}\alpha^{-1}\left(\frac{u}{e^u-1}-1\right)+\frac{1}{12}\left(\frac{1}{2}+\frac{1}{e^u-1}-\frac{ue^u}{(e^u-1)^2}\right)$$

$$= n-\frac{u}{\alpha}+O(\alpha^2) \qquad (41)$$

and

$$\int_0^u \frac{t}{e^t-1}\,dt = c^{-2}-\int_u^\infty \frac{t}{e^t-1}\,dt = c^{-2}+u\log(1-e^{-u})-\sum_{k=1}^{\infty}k^{-2}e^{-ku}$$

$$= c^{-2}-(u+1)e^{-u}+O(ue^{-2u}), \qquad (42)$$

one obtains

$$\alpha = c^{-1}n^{-\frac{1}{2}}-\tfrac{1}{2}(ce^{-\lambda}-1)un^{-1}-\tfrac{1}{2}(\tfrac{1}{2}+ce^{-\lambda})n^{-1}+O(n^{-\frac{1}{2}}\log^2 n).$$

Noting that $A_0 = 2c^{-2}+O(n^{-\frac{1}{2}})$, the right-hand side of (40) becomes

$$(e^{-\lambda}-c^{-1})n^{-\frac{1}{2}}-\tfrac{1}{4}ce^{-\lambda}u^2n^{-1}+(\tfrac{5}{4}ce^{-\lambda}-\tfrac{1}{2})un^{-1}+$$
$$+(1+\tfrac{1}{2}ce^{-\lambda}-c^{-1}e^{-\lambda}+\tfrac{1}{2}e^{-2\lambda})n^{-1}+O(n^{-\frac{1}{2}}\log^4 n).$$

This becomes zero only if $ce^{-\lambda} = 1+\mu$, $\mu = O(n^{-\frac{1}{2}}\log^2 n)$,

$$u = \tfrac{1}{2}\log n+\lambda = L-\mu+O(n^{-1}\log^4 n),$$

where $L = \log(cn^{\frac{1}{2}})$, i.e.

$$\mu c^{-1}n^{-\frac{1}{2}}-\tfrac{1}{4}L^2n^{-1}+\tfrac{3}{4}Ln^{-1}+(\tfrac{3}{2}-\tfrac{1}{2}c^{-2})n^{-1} = O(n^{-\frac{1}{2}}\log^4 n),$$
$$\mu = \tfrac{1}{4}cL^2n^{-\frac{1}{2}}-\tfrac{3}{4}cLn^{-\frac{1}{2}}-(\tfrac{1}{2}c^{-1}-\tfrac{3}{2}c)n^{-\frac{1}{2}}+O(n^{-1}\log^4 n),$$
$$\alpha = c^{-1}n^{-\frac{1}{2}}-\tfrac{3}{4}n^{-1}+O(n^{-\frac{1}{2}}\log^3 n),$$
$$k = k_1 = u\alpha^{-1} = cn^{\frac{1}{2}}L-c^2(\tfrac{1}{4}L^2-\tfrac{3}{2}L-\tfrac{3}{2})-\tfrac{1}{2}+O(n^{-\frac{1}{2}}\log^4 n).$$

If k is less than k_1, the right-hand expression in (40) is positive; if k is greater than k_1, the expression is negative. This proves Theorem 2 for $n \leqslant c_1 k^2$.

For $k = o(n^{\frac{1}{2}})$ one has to use (29) instead of (32). One gets

$$\Delta \log p(n,k) = -\log(1-e^{-u})+O(k^{-1}),$$

which is positive since $u = O(1)$.

The proof of Theorem 3 is very similar except that (31) and (41) must be replaced by

$$\sum_{\nu=1}^{k}\frac{\nu}{e^{\nu\alpha-1}} = n-\binom{k+1}{2}, \qquad \sum_{\nu=1}^{k+1}\frac{\nu}{e^{\nu(\alpha+\Delta\alpha)}-1} = n-\binom{k+2}{2}, \tag{43}$$

and

$$\alpha^{-2}\int_0^u \frac{t}{e^t-1}\,dt+\tfrac{1}{2}\alpha^{-1}\left(\frac{u}{e^u-1}-1\right)+\tfrac{1}{2}u^2\alpha^{-2}+\tfrac{1}{2}u\alpha^{-1} = n+O(1). \tag{44}$$

From (43), $$\Delta\alpha = A_0^{-1}\alpha^2\frac{ue^u}{e^u-1}+O(\alpha^3),$$

$$\Delta\left[\sum_{\nu=1}^{k}\left\{\frac{\nu\alpha}{e^{\nu\alpha}-1}-\log(1-e^{-\nu\alpha})\right\}\right]$$
$$= -\log(e^u-1)-\alpha\left\{\frac{e^u}{e^u-1}+\tfrac{1}{2}A_0^{-1}\frac{u^2e^{2u}}{(e^u-1)^2}\right\}+O(\alpha^2),$$

$$\tfrac{3}{2}\Delta\{\log\alpha\} = \tfrac{3}{2}A_0^{-1}\alpha\frac{ue^u}{e^u-1}+O(\alpha^2),$$

$$-\tfrac{1}{2}\Delta\{\log A_0\} = -\tfrac{1}{2}\alpha\left\{A_0^{-1}\frac{u^2e^u}{(e^u-1)^2}+A_0^{-2}\frac{u^4e^{2u}}{(e^u-1)^3}+O(\alpha^2),\right.$$

$$\Delta\{\alpha A_1(u)\} = O(\alpha^2).$$

Hence

$$\Delta \log q(n,k) = -\log(e^u-1) - \alpha\left\{\frac{e^u}{e^u-1} + \tfrac{1}{2}A_0^{-1}\frac{u^2(e^u+e^{2u})}{(e^u-1)^2} - \right.$$

$$\left. - \tfrac{3}{2}A_0^{-1}\frac{ue^u}{e^u-1} + \tfrac{1}{2}A_0^{-2}\frac{u^4e^{2u}}{(e^u-1)^3}\right\} + O(\alpha^2),$$

which is zero only if $u = \log 2 + \lambda n^{-\frac{1}{3}}$ (λ bounded).

Writing $b = c^2 \log^2 2$, we see from (44), in view of†

$$\int_0^{\log 2} \frac{t}{e^t-1}\,dt = \tfrac{1}{2}c^{-2}(1-b) \quad \text{and} \quad A_0 = c^{-2}(1-2b) + O(n^{-\frac{1}{3}}),$$

that $\alpha = c^{-1}(2n)^{-\frac{1}{3}} + (\frac{1}{2}\log 2 - \frac{1}{4} + \lambda c 2^{\frac{1}{3}} \log 2) n^{-1} + O(n^{-\frac{4}{3}})$.

Hence

$$\Delta \log q(n,k) = \left\{-2\lambda - 2^{-\frac{1}{3}}c^{-1}\left(2 + 3\,\frac{b}{1-2b} - \right.\right.$$

$$\left.\left. - \frac{3b}{\log 2\,(1-2b)} + \frac{2b^2}{(1-2b)^2}\right)\right\} n^{-\frac{1}{3}} + O(n^{-1}),$$

which is zero only if

$$\lambda = -2^{-\frac{1}{3}}c^{-1}\left[1 + \frac{3}{2}\,\frac{b}{1-2b}\{1-(\log 2)^{-1}\} + \frac{b^2}{(1-2b)^2}\right] + O(n^{-\frac{1}{3}}),$$

$$k = k_0 = u\alpha^{-1} = 2^{\frac{1}{3}}(c\log 2)n^{\frac{1}{3}} - c^2 \log 2(\log 2 - \tfrac{1}{2}) + 2^{\frac{1}{3}}c(1-2b)\lambda + O(n^{-\frac{1}{3}})$$

$$= 2^{\frac{1}{3}}(c\log 2)n^{\frac{1}{3}} - b + \tfrac{1}{2}b(\log 2)^{-1} - 1 +$$

$$+ 2b - \tfrac{3}{2}b\{1-(\log 2)^{-1}\} - \frac{b^2}{1-2b} + O(n^{-\frac{1}{3}})$$

$$= 2^{\frac{1}{3}}(c\log 2)n^{\frac{1}{3}} - 1 + 2b(\log 2)^{-1} - \frac{\frac{1}{2}b}{1-2b} + O(n^{-\frac{1}{3}}).$$

† D. Bierens de Haan, *Nouvelles tables d'intégrales définies* (New York, 1939), 151, formula (104, 5).

SOME CONGRUENCES FOR BERNOULLI NUMBERS OF HIGHER ORDER

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1. Introduction

THE Bernoulli numbers of order k may be defined [(**4**), Chapter 6], as

$$\left(\frac{x}{e^x-1}\right)^k = \sum_{m=0}^{\infty} \frac{x^m}{m!} B_m^{(k)} \quad (|x| < 2\pi);$$

in particular $B_m^{(1)} = B_m$, the ordinary Bernoulli number. S. Wachs (**5**) proved a result equivalent to

$$B_{p+2}^{(p+1)} \equiv 0 \pmod{p^2}, \tag{1.1}$$

where p is a prime not less than 3. The writer (**1**) improved (1.1) to

$$B_{p+2}^{(p+1)} \equiv 0 \pmod{p^3} \quad (p \geqslant 5); \tag{1.2}$$

the residue of $B_{p+2}^{(p+1)} \pmod{p^4}$ was not determined. In the present paper I show that

$$B_{p+2}^{(p+1)} \equiv \tfrac{1}{6}p^3 \pmod{p^4} \quad (p \geqslant 5). \tag{1.3}$$

In (**1**) it was also proved that

$$B_p^{(p)} \equiv \tfrac{1}{2}p^2 \pmod{p^3} \quad (p \geqslant 3) \tag{1.4}$$

and indeed that

$$B_p^{(p)} \equiv \tfrac{1}{2}p^2\left(B_{p-1} + \frac{1}{p} - (p+1)K_p\right) \pmod{p^4}, \tag{1.5}$$

where

$$K_p = \sum_{a=1}^{p-1} k_a, \qquad r^{p-1} - 1 = pk_r \quad (p \nmid r).$$

I shall improve this to

$$B_p^{(p)} \equiv -\tfrac{1}{2}p^2(p-1)! \pmod{p^5} \tag{1.6}$$

for $p \geqslant 5$.

The proof of (1.5) and (1.6) makes use of the following two results which were not used in (**1**): first Glaisher's formula [(**2**) § 19]

$$A_{2t+1}^{(p)} \equiv \tfrac{1}{2}p(p-2t-1)A_{2t}^{(p)} \pmod{p^4} \quad (t > 1;\ p > 5), \tag{1.7}$$

where

$$(x+1)(x+2)\ldots(x+m) = \sum_{r=0}^{m} A_r^{(m+1)} x^{m-r}, \tag{1.8}$$

and second Nörlund's formula [(**4**), 149]

$$(B+1)(B+2)\ldots(B+m) = \frac{m!}{m+1}; \tag{1.9}$$

it is understood that, after expansion of the left member of (1.9), B^r is replaced by B_r.

For the convenience of the reader the proofs in the present paper are complete in themselves; knowledge of (1) is not assumed. Throughout p denotes a prime number exceeding 5.

2. In the formula [(4) 148]

$$B_m^{(k+1)} = (k+1)\binom{m}{k+1}\sum_{s=0}^{k}(-1)^{k-s}\binom{k}{s}B_s^{(k+1)}\frac{B_{m-s}}{m-s}, \tag{2.1}$$

take $k = p$, $m = p+2$; then (2.1) becomes

$$B_{p+2}^{(p+1)} = (p+1)(p+2)\sum_{s=0}^{p}(-1)^{p-s}\binom{p}{s}B_s^{(p+1)}\frac{B_{p+2-s}}{p+2-s}. \tag{2.2}$$

Since [by (4) 148]

$$(-1)^s\binom{m}{s}B_s^{(m+1)} = A_s^{(m+1)}, \tag{2.3}$$

where $A_s^{(m+1)}$ is defined by (1.8), (2.2) may be reduced to

$$B_{p+2}^{(p+1)} = -(p+1)(p+2)\Big(u_1+u_3+\sum_{t=2}^{\frac{1}{2}(p-3)}u_{2t+1}+u_p\Big), \tag{2.4}$$

where

$$u_{2t+1} = A_{2t+1}^{(p+1)}\frac{B_{p+1-2t}}{p+1-2t} \quad (0 \leqslant t \leqslant \tfrac{1}{2}(p-1)). \tag{2.5}$$

In view of (1.7),

$$A_{2t+1}^{(p+1)} = A_{2t+1}^{(p)}+pA_{2t}^{(p)} \equiv \tfrac{1}{2}p(p-2t+1)A_{2t}^{(p)} \pmod{p^4};$$

consequently

$$u_{2t+1} \equiv \tfrac{1}{2}pA_{2t}^{(p)}B_{p+1-2t} \pmod{p^4} \quad (2 \leqslant t \leqslant \tfrac{1}{2}(p-3)),$$

which implies

$$S = \sum_{t=2}^{\frac{1}{2}(p-3)}u_{2t+1} \equiv \tfrac{1}{2}p\sum_{t=2}^{\frac{1}{2}(p-3)}A_{2t}^{(p)}B_{p+1-2t} \pmod{p^4}. \tag{2.6}$$

3. Consider now the sum

$$\sum_{r=0}^{p-1}A_r^{(p)}B_{p+1-r} = B^2(B+1)(B+2)\ldots(B+p-1). \tag{3.1}$$

Put $(B)_m = (B+1)(B+2)\ldots(B+m)$; then clearly

$$\begin{aligned} B(B+1)\ldots(B+m-1) &= (B)_m - m(B)_{m-1} \\ &= \frac{m!}{m+1} - m\frac{(m-1)!}{m} = -\frac{(m-1)!}{m+1}, \end{aligned}$$

by (1.9), and

$$\begin{aligned} B^2(B+1)\ldots(B+m-1) &= B(B+1)\ldots(B+m) - mB(B+1)\ldots(B+m-1) \\ &= -\frac{m!}{m+2} + m\frac{(m-1)!}{m+1} = \frac{m!}{(m+1)(m+2)}. \end{aligned}$$

Thus (3.1) becomes

$$B_{p+1}+A_2^{(p)}B_{p-1}+\sum_{t=2}^{\frac{1}{2}(p-3)}A_{2t}^{(p)}B_{p+1-2t}+(p-1)!\,B_2=\frac{p!}{(p+1)(p+2)}. \tag{3.2}$$

Comparison of (3.2) with (2.4) and (2.6) yields

$$B_{p+2}^{(p+1)}\equiv -(p+1)(p+2)\times$$
$$\times\left\{u_1+u_3+u_p-\tfrac{1}{2}p(B_{p+1}+A_2^{(p)}B_{p-1}+(p-1)!\,B_2)+\tfrac{1}{2}p\,\frac{p!}{(p+1)(p+2)}\right\} \pmod{p^4}. \tag{3.3}$$

Next we recall the special values [(**2**) § 46]

$$\left.\begin{aligned} A_1^{(m)}&=\tfrac{1}{2}m(m-1)\\ A_2^{(m)}&=\tfrac{1}{24}m(m-1)(m-2)(3m-1)\\ A_3^{(m)}&=\tfrac{1}{48}m^2(m-1)^2(m-2)(m-3)\end{aligned}\right\}. \tag{3.4}$$

Thus
$$u_1=\tfrac{1}{2}pB_{p+1},\ u_p=(1/12)p!,$$
$$u_3=A_3^{(p+1)}\frac{B_{p-1}}{p-1}=\tfrac{1}{48}(p+1)^2p^2(p-2)B_{p-1}.$$

Direct substitution in the right member of (3.3) now yields

$$-(p+1)(p+2)\times$$
$$\times\left\{\tfrac{1}{48}p^2(p-2)[(p+1)^2-(p-1)(3p-1)]B_{p-1}+\tfrac{1}{2}p\,\frac{p!}{(p+1)(p+2)}\right\}.$$

Since [(**3**) § 14]

$$pB_{p-1}\equiv(p-1)!+p \pmod{p^2}, \tag{3.5}$$

a little manipulation leads to

$$B_{p+2}^{(p+1)}\equiv\tfrac{1}{6}p^3 \pmod{p^4}. \tag{3.6}$$

As for the excluded case $p=5$, we have

$$B_7^{(6)}=-\frac{1375}{4}=-\frac{11\cdot5^3}{4}\equiv\frac{1}{6}5^3 \pmod{5^4},$$

so that (3.6) holds for $p=5$. This completes the proof of (1.3). I remark that $B_5^{(4)}=-9$.

4. Returning to (2.1), we take $k+1=m=p$, so that

$$B_p^{(p)}=p\sum_{s=0}^{p-1}(-1)^s\binom{p-1}{s}B_s^{(p)}\frac{B_{p-s}}{p-s}$$
$$=p\left\{\tfrac{1}{2}pB_{p-1}-\tfrac{1}{2}(p-1)!+\sum_{t=2}^{\frac{1}{2}(p-3)}A_{2t+1}^{(p)}\frac{B_{p-1-2t}}{p-1-2t}+A_3^{(p)}\frac{B_{p-3}}{p-3}\right\}. \tag{4.1}$$

By (1.7),

$$S = \sum_{t=2}^{\frac{1}{2}(p-3)} A_{2t+1}^{(p)} \frac{B_{p-1-2t}}{p-1-2t} \equiv \tfrac{1}{2}p \sum_{t=2}^{\frac{1}{2}(p-3)} A_{2t}^{(p)} B_{p-1-2t} \pmod{p^4}. \quad (4.2)$$

Again, as above,

$$\sum_{s=0}^{p-1} A_s^{(p)} B_{p-1-s} = (B+1)(B+2)\ldots(B+p-1) = \frac{(p-1)!}{p},$$

which we break up into

$$B_{p-1}+(p-1)!-\tfrac{1}{2}A_{p-2}^{(p)}+\sum_{2}^{\frac{1}{2}(p-3)} A_{2t}^{(p)} B_{p-1-2t}+A_2^{(p)}B_{p-3} = \frac{(p-1)!}{p}. \quad (4.3)$$

Comparing with (4.2) we get

$$S \equiv \tfrac{1}{2}(p-1)!-\tfrac{1}{2}p!-\tfrac{1}{2}pB_{p-1}+\tfrac{1}{4}pA_{p-2}^{(p)}-\tfrac{1}{2}pA_2^{(p)}B_{p-3} \pmod{p^4}.$$

We next substitute in the extreme right member of (4.1). Thus the expression within $\{\}$ becomes

$$-\tfrac{1}{2}p!+\tfrac{1}{4}pA_{p-2}^{(p)}+\left\{\frac{1}{p-3}A_3^{(p)}-\tfrac{1}{2}pA_2^{(p)}\right\}B_{p-3}. \quad (4.4)$$

To simplify (4.4) we use (3.4) and Glaisher's formula [(**3**) § 14],

$$A_{p-2}^{(p)} \equiv p^2 \frac{p-2}{2(p-3)} B_{p-3} \pmod{p^3},$$

which implies $\quad \tfrac{1}{4}pA_{p-2}^{(p)} \equiv \tfrac{1}{12}p^3B_{p-3} \pmod{p^4}.$

Thus we find without much trouble that the right member of (4.4) is congruent to $-\tfrac{1}{2}p(p-1)! \pmod{p^4}$. Hence (4.1) becomes

$$B_p^{(p)} \equiv -\tfrac{1}{2}p^2(p-1)! \pmod{p^5}. \quad (4.5)$$

Here again (4.5) has been proved for $p > 5$. As for $p = 5$, we have

$$B_5^{(5)} = -\frac{475}{12} = -\frac{19.5^2}{12} \equiv -\frac{1}{2}.5^2.4! \pmod{5^5},$$

so that (4.5) holds for $p = 5$ also. This completes the proof of (1.6).

5. It is perhaps not evident that (1.6) implies (1.5). To show that this is the case it suffices to prove

$$-(p-1)! \equiv B_{p-1}+\frac{1}{p}-(p+1)K_p \pmod{p^2}. \quad (5.1)$$

Now from the definition of K_p, we have

$$pK_p = \sum_{r=1}^{p-1}(r^{p-1}-1) = \frac{1}{p}B_p(p)-(p-1), \quad (5.2)$$

where $B_p(x)$ denotes the Bernoulli polynomial of degree p,

$$B_p(x) = \sum_{s=0}^{p} \binom{p}{s} B_s x^{p-s}.$$

Thus (5.2) implies

$$K_p \equiv B_{p-1} + \frac{1}{p} - 1 \pmod{p^2}, \tag{5.3}$$

so that

$$(p+1)K_p \equiv (p+1)B_{p-1} + \frac{1}{p} - p \equiv (p-1)! + B_{p-1} + \frac{1}{p} \pmod{p^2},$$

by (3.5). This evidently proves (5.1).

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ORBITAL TOPOLOGIES

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1. Introduction

A NATURAL question in general topology is: *Given a set S and a mapping γ of S into itself, how may one construct a topology in S, in general non-trivial, so that in this topology γ will be continuous?* In this note I show how to construct a topology from any such pair S and γ and show that, if γ is finite-to-one, γ is continuous in this topology and that, if γ is bi-uniform and onto S, then γ is actually a homeomorphism of S onto itself in this topology. In the last section of the note I find as an application a theorem asserting the existence of T_1 spaces having unusual groups of homeomorphisms.

2. The space S_γ

Let S be a set and γ a mapping of S into itself. The set $\{x\gamma^n\}$ ($n = 0, 1, 2, \ldots$) for x in S is called the *orbit* of x under γ and written $\Gamma(x)$. For $X \subset S$ and $x \in S$ define: $x \in X'$ if and only if $x \in \Gamma(y)$ for infinitely many distinct y in X. As usual, then, one defines $\overline{X} = X \cup X'$, and other topological notions such as *open set*, *closed set*, etc., are defined in terms of closure $\overline{X}$ and derived set X'. The resulting topological space will be denoted by S_γ and the topology may be called the 'orbital topology' of S under γ.

Remark. S_γ is discrete if and only if $\bigcap_{x \in E} \Gamma(x)$ is null for every infinite set E in S. S_γ may be even dense-in-itself, as is shown by taking $S = (0, 1)$ and $x\gamma = x^2$.

THEOREM 1. *S_γ is a Kuratowski space in which $(X \cup Y)' = X' \cup Y'$. S_γ is a T_1 space but a necessary condition that it be a T_2 (Hausdorff) space is that every accumulation point be a fixpoint under γ.*

Proof. From the definitions of X' and $\overline{X}$, it is obvious that $\overline{\emptyset} = \emptyset$, $X \subset \overline{X}$, $(X \cup Y)' = X' \cup Y'$, and $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$. To show that $\overline{\overline{X}} \subset \overline{X}$ let $p \in \overline{\overline{X}}$ and suppose p not in $\overline{X}$. Then p is in $\Gamma(y)$ for infinitely many y in $\overline{X} - X$. But any such y is in $\Gamma(z)$ for infinitely many z in X. Hence a contradiction follows.

Since finite sets are closed, S_γ is T_1. If there is an accumulation point not fixed under γ, then limits of sequences are not unique and the space cannot be a Hausdorff space.

THEOREM 2. *No orbit possesses an accumulation point in S_γ (the term 'orbit' means, of course, 'orbit under γ').*

Proof. Suppose that $p \in \Gamma'(x)$. Then for infinitely many n there are i_n such that $x\gamma^n\gamma^{i_n} = p$. Infinitely many i_n+n must be distinct and γ is periodic at p, so that $\Gamma(x)$ is finite and a contradiction is reached.

Thus orbits are closed. Although arbitrary set-theory sums of closed sets are not generally closed in a Kuratowski space, sums of orbits enjoy this property.

THEOREM 3. *Any set-theory sum of orbits in S is closed.*

Proof. Suppose $$x \in \overline{\bigcup_{y \in X} \Gamma(y)}.$$

Then $x \in \Gamma(z)$ for some z in $\bigcup_{y \in X} \Gamma(y)$. But this implies

$$x \in \bigcup_{y \in X} \Gamma(y).$$

Remark. Since no orbit has an accumulation point in S_γ, a necessary condition that S_γ be compact (countably compact) is that all orbits be finite.

3. γ finite-to-one

THEOREM 4. *If γ is finite-to-one, then γ is continuous in S_γ.*

Proof. I shall show that, if A is open, then $A\gamma^{-1}$ (actually an abbreviation for $(A \cap S_\gamma)\gamma^{-1}$) is open. Suppose A open and $A\gamma^{-1}$ not open. Then there are infinitely many distinct points $x_1, x_2, \ldots$ in $C(A\gamma^{-1})$, where $C(\ )$ denotes complement in S, and a point q in $A\gamma^{-1}$ with $q \in \Gamma(x_n)$ for all n. But $q\gamma \in \Gamma(x_n\gamma)$ for all n, $x_n\gamma \notin A$ for all n, $q\gamma \in A$, and the set $\{x_n\gamma\}$ is infinite since γ is finite-to-one. Thus, $C(A)$ is not closed and a contradiction is reached.

Remark. I have succeeded neither in proving nor disproving Theorem 4 without the finite-to-one hypothesis.

A more fascinating result is

THEOREM 5. *If γ is an arbitrary mapping and δ is a finite-to-one mapping of S into itself and $\delta\gamma = \gamma\delta$ on S, then δ is continuous in S_γ.*

Proof. Let A be an open set in S_γ and again suppose $A\delta^{-1}$ not open. Then there is $q \in A\delta^{-1}$ and an infinite set of distinct elements $x_1, x_2, \ldots$ in $C(A\delta^{-1})$ with $x_n\gamma^{i_n} = q$ for all n. Then $x_n\gamma^{i_n}\delta = q\delta$ and $(x_n\delta)\gamma^{i_n} = q\delta$ for all n. Thus $q\delta$ is in $\Gamma(x_n\delta)$ for all n. Since δ is finite-to-one, the set $\{x_n\delta\}$ is infinite and it lies in $C(A)$. But $q\delta$ is in A which contradicts the openness of A.

We have, almost as a corollary to Theorem 5,

THEOREM 6. *If γ is bi-uniform and onto, it is a homeomorphism of S_γ onto itself.*

Proof. From Theorem 4, γ is continuous. Also, γ^{-1} is finite-to-one and commutes with γ on S so that γ^{-1} is continuous by Theorem 5.

4. An application

Let G be an additively written Abelian group. For $a \in G$ denote the right translation of G defined by a by ρ_a. We abbreviate $G\rho_a$ to G_a.

THEOREM 7. *G is isomorphic to a group of homeomorphisms of G_a onto itself. In G_a, the group operation is continuous in each variable separately.*

Proof. G is isomorphic to its (right) regular representation as a permutation group. By Theorem 6, ρ_a is a homeomorphism of G_a onto itself, and it follows from Theorem 5 that each (right) translation is a homeomorphism of G onto itself. For the last part of the conclusion of Theorem 7, we need merely verify as in the proof of Theorem 5 that any translation is an open mapping.

As a final novelty I establish

THEOREM 8. *Let $\aleph_j$ be any infinite cardinal. There is a dense-in-itself T_1 space of cardinal $2^{\aleph_j}$ whose homeomorphism group contains an Abelian subgroup of cardinal $2^{\aleph_j}$ of involutory homeomorphisms.*

Proof. Select a set of cardinal $\aleph_j$. Let A be the additive group of the Boolean ring associated to the Boolean algebra of all subsets of the selected set. Let B be an infinite cyclic group generated by a. In the direct product $A \otimes B$, any element is in the $\rho_{(0,a)}$ orbit of infinitely many distinct elements. Thus, $(A \otimes B)_{(0,a)}$ is a dense-in-itself T_1 space and A is the required group of involutory homeomorphisms since each element of A is nilpotent.

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ON A GENERALIZATION OF PÓLYA'S RANDOM-WALK THEOREM

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1. We consider a particle which moves on the points of an n-dimensional lattice. Let the points be labelled by vectors,

$$\mathbf{x} = (x_1, x_2, \ldots, x_n),$$

with integral components, and suppose that the particle in position $\mathbf{x}$ has a probability $f(\mathbf{y})$ of jumping in one step to position $\mathbf{x}+\mathbf{y}$, where $f(\mathbf{y})$ is independent of $\mathbf{x}$. Further, suppose that the probability distribution $\{f(\mathbf{x})\}$ is symmetrical (about the origin):

$$f(\mathbf{x}) = f(-\mathbf{x}) \text{ for all } \mathbf{x}.$$

Define

$$\mathbf{0} \equiv (0, 0, \ldots, 0),$$

$$||\mathbf{x}|| \equiv |x_1| + |x_2| + \ldots + |x_n|,$$

$$\mathbf{u}_j \equiv (0, 0, \ldots, 1, \ldots, 0),$$

where the jth component is 1.

A point is called *recurrent* if there is probability unity that the particle, initially at that point, will sooner or later return to it; otherwise it is called *transient* [cf. (**2**) 320]. In the particular case where

$$f(\mathbf{x}) = \begin{cases} 1/2n, \text{ when } \mathbf{x} = \pm\mathbf{u}_j \ (j = 1, 2, \ldots, n), \\ 0 \text{ otherwise}, \end{cases}$$

Pólya (**3**) proved that *for* $n = 1$ *or* 2, *the points are recurrent, but for* $n \geqslant 3$ *they are transient.* We shall prove† that this theorem remains true for a wide class of symmetrical distributions $\{f(\mathbf{x})\}$. For $n \geqslant 3$ there is no exception to Pólya's result that the points are transient. An example is given, however, which shows that, even when $n = 1$, there exist distributions such that the points are transient. In §§ 2, 3 we treat the n-dimensional random walk, and in § 4 we consider in greater detail the one-dimensional case. Our main result is in § 4, where we find a sufficient condition and also a necessary condition for recurrence.

† Similar results have been obtained by Chung and Fuchs (**1**). Being interested in a more general problem, these authors obtain less precise results.

2. We use the characteristic function,

$$C(\boldsymbol{\theta}) \equiv \sum_{\mathbf{x}} f(\mathbf{x}) e^{i\mathbf{x}.\boldsymbol{\theta}},$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$. On account of the symmetry of $\{f(\mathbf{x})\}$, we have

$$C(\boldsymbol{\theta}) = \sum_{\mathbf{x}} f(\mathbf{x}) \cos(\mathbf{x}.\boldsymbol{\theta}).$$

When (as will be the case in the sequel) the components of $\boldsymbol{\theta}$ are real, $C(\boldsymbol{\theta})$ is real and $-1 \leqslant C(\boldsymbol{\theta}) \leqslant 1$. Moreover, $C(\boldsymbol{\theta})$ is continuous since the series defining it is absolutely and uniformly convergent.

Let $p(t)$ be the probability that the particle returns to its initial position at the t-th step (not necessarily for the first time). Then $p(t)$ is the term independent of $\boldsymbol{\theta}$ in the Fourier expansion in $(0, 2\pi)$ of $\{C(\boldsymbol{\theta})\}^t$. Therefore

$$p(t) = (2\pi)^{-n} \int_0^{2\pi} \cdots \int_0^{2\pi} \{C(\boldsymbol{\theta})\}^t \, d\boldsymbol{\theta}. \tag{2.1}$$

By a known theorem [cf. (**2**) 244], the probability that the particle never returns to its initial position is

$$1 \Big/ \sum_{t=0}^{\infty} p(t), \tag{2.2}$$

a necessary and sufficient condition for recurrence thus being that

$$\sum_{t=0}^{\infty} p(t) = \infty. \tag{2.3}$$

[When (2.3) is true, (2.2) is to be interpreted as zero.] We prove now that the probability (2.2) is equal to

$$(2\pi)^n \Big/ \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{d\boldsymbol{\theta}}{1 - C(\boldsymbol{\theta})}. \tag{2.4}$$

We first observe that

$$\sum_{t=0}^{N} p(t) = (2\pi)^{-n} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{1 - \{C(\boldsymbol{\theta})\}^{N+1}}{1 - C(\boldsymbol{\theta})} \, d\boldsymbol{\theta}.$$

$C(\boldsymbol{\theta})$ attains its maximum value of unity only on a set of measure zero in the domain of integration. [We exclude the trivial case of no motion when $C(\boldsymbol{\theta}) \equiv 1$.] Therefore the integrand is non-negative and, for almost all $\boldsymbol{\theta}$, tends monotonically to the limit $1/\{1 - C(\boldsymbol{\theta})\}$ as $N \to \infty$ through even values or through odd values. The desired result (2.4) follows by a well-known property of Lebesgue integrals [cf. (**4**) 28].

We see then that the divergence of the integral

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \frac{d\boldsymbol{\theta}}{1-C(\boldsymbol{\theta})} \tag{2.5}$$

is necessary and sufficient for recurrence.

We may always suppose that the given distribution $\{f(\mathbf{x})\}$ is genuinely an n-dimensional distribution. For, if the particle were restricted (with probability unity) to the points of a sub-lattice, we could redefine the problem for this sub-lattice. $C(\boldsymbol{\theta})$ attains its maximum value of unity at $\boldsymbol{\theta} = \mathbf{0}$. Further, there are only a finite number of values of $\boldsymbol{\theta}$ in the domain of integration where $C(\boldsymbol{\theta}) = 1$. For let $\boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \ldots, \boldsymbol{\nu}_n$ be n independent vectors, with integral components, such that

$$f(\boldsymbol{\nu}_r) > 0 \quad (r = 1, 2, \ldots, n). \tag{2.6}$$

Then, if $C(\boldsymbol{\varphi}) = 1$,

$$\boldsymbol{\nu}_r . \boldsymbol{\varphi} = 2\lambda_r \pi \quad (r = 1, 2, \ldots, n)$$

for some integers λ_r. For fixed integers λ_r, this system of equations has a unique solution $\boldsymbol{\varphi}$, and there are only a finite number of such systems of equations (obtained by varying the integers λ_r) which have a solution $\boldsymbol{\varphi}$ within the domain of integration. In fact we must have

$$|\lambda_r| \leqslant ||\boldsymbol{\nu}_r|| \quad (r = 1, 2, \ldots, n).$$

Near each of these isolated values $\boldsymbol{\varphi}$ the integrand in (2.5) behaves in the same way since each $\boldsymbol{\varphi}$ is a period of $C(\boldsymbol{\theta})$. Therefore the integral (2.5) diverges if and only if

$$\int_0^{\alpha} \cdots \int_0^{\alpha} \frac{d\boldsymbol{\theta}}{1-C(\boldsymbol{\theta})} = \infty \tag{2.7}$$

for all sufficiently small α.

By Jordan's inequality, we have, for $-\pi \leqslant \beta \leqslant \pi$,

$$1-\cos\beta = 2\sin^2(\tfrac{1}{2}\beta) \geqslant 2\beta^2/\pi^2.$$

Therefore

$$1-C(\boldsymbol{\theta}) \geqslant 2\pi^{-2} \sum_{r=1}^{n} f(\boldsymbol{\nu}_r)(\boldsymbol{\nu}_r . \boldsymbol{\theta})^2,$$

provided that $|\boldsymbol{\nu}_r . \boldsymbol{\theta}| \leqslant \pi$ for each r, i.e. that $||\boldsymbol{\theta}||$ is small enough. By (2.6) the expression on the right is a positive definite form in $\theta_1, \theta_2, \ldots, \theta_n$. Therefore

$$1-C(\boldsymbol{\theta}) \geqslant K(\theta_1^2 + \ldots + \theta_n^2),$$

where† K (>0) may be taken as the smallest latent root associated with the quadratic form. Therefore, for all sufficiently small α,

$$\int_0^\alpha \dots \int_0^\alpha \frac{d\boldsymbol{\theta}}{1-C(\boldsymbol{\theta})} \leqslant K^{-1} \int_0^\alpha \dots \int_0^\alpha \frac{d\theta_1 \dots d\theta_n}{\theta_1^2+\dots+\theta_n^2}.$$

The integral on the right diverges for $n = 1, 2$, but converges for $n \geqslant 3$. It follows that in the random walk of three or more dimensions, for any symmetrical distribution $\{f(\mathbf{x})\}$, the points are always transient. The cases of one and two dimensions, however, require further investigation.

3. Under a rather weak assumption, a generalization of Pólya's theorem can be proved by a method which gives insight into the 'real reason' for the truth of the theorem. Our assumption, made only in the present section, is that

$$\sum_{\mathbf{x}} ||\mathbf{x}||^2 f(\mathbf{x}) < \infty.$$

It follows that all first and second partial derivatives

$$\frac{\partial C}{\partial \theta_j}, \quad \frac{\partial^2 C}{\partial \theta_j \partial \theta_k}$$

exist and are continuous since, for example,

$$\left|\frac{\partial^2 C}{\partial \theta_j \partial \theta_k}\right| \leqslant \sum_{\mathbf{x}} |x_j x_k| f(\mathbf{x}) \leqslant \sum_{\mathbf{x}} ||\mathbf{x}||^2 f(\mathbf{x}).$$

An n-dimensional analogue of Laplace's asymptotic formula [cf. (5) 277] shows that, for any sufficiently small α,

$$\int_0^\alpha \dots \int_0^\alpha \{C(\boldsymbol{\theta})\}^t \, d\boldsymbol{\theta} \sim K t^{-\frac{1}{2}n} \quad \text{as } t \to \infty.$$

It follows that

$$p(t) \sim K t^{-\frac{1}{2}n} \quad \text{as } t \to \infty.$$

Therefore the series in (2.3) diverges when $n = 1, 2$, and converges when $n \geqslant 3$, and Pólya's theorem is proved (and generalized).

4. Finally we consider in greater detail the one-dimensional case. We shall obtain a sufficient condition and a necessary condition for recurrence, i.e., for the divergence of the integral

$$\int_0^\alpha \frac{d\theta}{1-C(\theta)}, \tag{4.1}$$

where α is positive and arbitrarily small.

† In the sequel K will indicate a positive absolute constant, but not necessarily the same constant each time it occurs.

Define $F(r) \equiv \sum_{x=r}^{\infty} f(x)$. Then

$$\begin{aligned}
\tfrac{1}{2}\{1-C(\theta)\} &= \tfrac{1}{2}\sum_{-\infty}^{\infty} f(x)(1-\cos x\theta) \\
&= \sum_{1}^{\infty} f(x)(1-\cos x\theta), \text{ by evenness,} \\
&= 2\sum_{1}^{\infty} f(x)\sin^2(\tfrac{1}{2}x\theta) \\
&\leqslant \tfrac{1}{2}\theta^2 \sum_{1}^{M} x^2 f(x) + 2\sum_{M+1}^{\infty} f(x), \text{ for any } M, \\
&= \tfrac{1}{2}\theta^2\Big\{\sum_{1}^{M}(2x-1)F(x) - M^2F(M+1)\Big\} + 2F(M+1) \\
&\leqslant \theta^2 \sum_{1}^{M} xF(x), \quad \text{if } M|\theta| \geqslant 2.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\tfrac{1}{2}\{1-C(\theta)\} &= 2\sum_{1}^{\infty} f(x)\sin^2(\tfrac{1}{2}x\theta) \\
&\geqslant 2\pi^{-2}\theta^2 \sum_{1}^{M} x^2 f(x), \quad \text{if } M|\theta| \leqslant \pi.
\end{aligned}$$

Define

$$V(r) \equiv \sum_{1}^{r} xF(x),$$

$$v(r) \equiv \sum_{1}^{r} x^2 f(x),$$

and

$$M_\theta \equiv [\pi|\theta|^{-1}],$$

the greatest integer not exceeding $\pi|\theta|^{-1}$. Then, when $|\theta| < \pi-2$,

$$2 \leqslant M_\theta|\theta| \leqslant \pi,$$

and so, by the above,

$$4\pi^{-2}v(M_\theta) \leqslant \{1-C(\theta)\}/\theta^2 \leqslant 2V(M_\theta).$$

Both $v(r)$ and $V(r)$ are non-decreasing with r, and we can define the step-functions

$$w(\theta^{-1}) \equiv v(M_\theta),$$

$$W(\theta^{-1}) \equiv V(M_\theta).$$

Then, for $|\theta| < \pi-2$,

$$4\pi^{-2}w(\theta^{-1}) \leqslant \{1-C(\theta)\}/\theta^2 \leqslant 2W(\theta^{-1}).$$

Therefore, for sufficiently small α,

$$\int_0^\alpha \frac{d\theta}{1-C(\theta)} \geqslant \frac{1}{2}\int_0^\alpha \frac{d\theta}{\theta^2 W(\theta^{-1})} = \frac{1}{2}\int_{\alpha^{-1}}^\infty \frac{dt}{W(t)},$$

$$\int_0^\alpha \frac{d\theta}{1-C(\theta)} \leqslant \tfrac{1}{4}\pi^2 \int_0^\alpha \frac{d\theta}{\theta^2 w(\theta^{-1})} = \tfrac{1}{4}\pi^2 \int_{\alpha^{-1}}^\infty \frac{dt}{w(t)}.$$

As $t \to \infty$, w^{-1} and W^{-1} tend monotonically to zero or to positive limits: so the integral test can be applied, and (4.1) diverges if

$$\sum_1^\infty \{V(r)\}^{-1} = \infty, \tag{4.2}$$

and only if

$$\sum_N^\infty \{v(r)\}^{-1} = \infty \tag{4.3}$$

for every N.

The condition (4.2) is satisfied in the interesting special case when the mean absolute step,

$$2\sum_1^\infty xf(x),$$

is finite. This condition was obtained in (**1**) and is less restrictive than that provisionally assumed in § 3 of the present paper.

We can now prove, for example, that we have recurrence when

$$f(x) \equiv K \log|x| \operatorname{loglog}|x|/x^2 \quad (|x| > 2).$$

(K is a normalizing constant.) For this distribution (4.2) is satisfied since

$$F(r) < K \log r \operatorname{loglog} r/(r-1) \quad (r > 2),$$

and

$$V(r) < Kr \log r \operatorname{loglog} r \qquad (r > 2).$$

On the other hand, if

$$f(x) \equiv K(\log|x|)^{1+\epsilon}/x^2 \quad (|x| > 0),$$

we have transience. For

$$v(r) \geqslant K\sum_1^r (\log x)^{1+\epsilon} \geqslant Kr(\log r)^{1+\epsilon},$$

so that (4.3) is not satisfied.

The question whether (4.2) and (4.3) are equivalent remains open. Thus we should like to find a distribution $\{f(x)\}$ which satisfies (4.3) but not (4.2). In any case a sufficient condition for their equivalence is that, for some $\epsilon > 0$ and for all sufficiently large r,

$$F(r+1) \leqslant (2-\epsilon)r^{-2}\sum_1^r xF(x). \tag{4.4}$$

For we have

$$\begin{aligned} v(r) &= \sum_1^r x^2\{F(x)-F(x+1)\} \\ &= \sum_1^r (2x-1)F(x)-r^2F(r+1) \\ &= 2V(r)-\sum_1^r F(x)-r^2F(r+1) \\ &\geqslant \epsilon V(r)-\sum_1^r F(x). \end{aligned}$$

We want to show that (4.3) implies (4.2), so we may assume that $V(r)\to\infty$ as $r\to\infty$. Then we see that

$$\sum_1^r F(x) = o\{V(r)\} \tag{4.5}$$

(see below), so that $\{v(r)\}^{-1} \leqslant K\{V(r)\}^{-1}$,

and the desired result follows.

To prove (4.5) note first that

$$V(r) \geqslant \sum_1^s xF(x)+s\sum_{s+1}^r F(x).$$

Given $\eta > 0$, take $s > 2/\eta$ and then take r so large that $r > s$ and

$$\sum_{s+1}^r F(x) > \sum_1^s F(x).$$

[This will be possible if $\sum F(x)$ is divergent; otherwise (4.5) would be trivial.] Then

$$\sum_1^r F(x) < 2\sum_{s+1}^r F(x) < 2s^{-1}V(r) < \eta V(r),$$

and (4.5) is proved.

The condition (4.4) is rather weak since it is deducible from the condition that, for some constant $\rho < 1$,

$$\limsup_{r\to\infty} F(r)/F([\rho r]) < 1. \tag{4.6}$$

The proof that (4.6) implies (4.4) is quite simple and is omitted.

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TWO DOUBLE HYPERGEOMETRIC INTEGRALS

By L. J. SLATER (*Cambridge*)

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1. Introduction

In previous papers (**1**), (**2**) I deduced general transformations of hypergeometric functions from single contour integrals of the Barnes type. In this paper I give two further integrals. The first is a double contour integral, from which is deduced a general transformation of double hypergeometric series. The second integral is the basic analogue of the first, and leads to the corresponding result for double basic series.

In the usual notation let

$$(a)_0 = 1, \qquad (a)_n = a(a+1)...(a+n-1).$$

It will be assumed always that there are A of the a parameters, B of the b parameters, and so on. Also

$$\Gamma[(a);(b)] = \frac{\Gamma(a_1)\Gamma(a_2)...\Gamma(a_A)}{\Gamma(b_1)\Gamma(b_2)...\Gamma(b_B)},$$

and a dash will denote the omission of a vanishing factor in a sequence, thus $(a)'-a_r$ denotes the sequence

$$a_1-a_r,...,a_{r-1}-a_r,a_{r+1}-a_r,...,a_A-a_r.$$

Let

$$\Gamma(s,t) = \Gamma\begin{bmatrix}(a)+s,(b)-s,(c)+t,(d)-t\\(e)+s,(f)-s,(g)+t,(h)-t,(j)+s+t,(k)-s-t\end{bmatrix}$$

and consider the integral

$$-4\pi^2 I_1 = \int\int \Gamma(s,t)\,dsdt$$

round a semicircular contour $ABCD$, to the right of the imaginary axis in the s-plane, and a similar contour $A'B'C'D'$ in the t-plane. Then

$$\begin{aligned}-4\pi^2 I_1 = &-\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\Gamma(Re^{i\theta},Re^{i\phi})R^2e^{i(\theta+\phi)}\,d\theta d\phi -\\ &-\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-iR}^{iR}\Gamma(s,Re^{i\phi})Rie^{i\phi}\,dsd\phi -\\ &-\int_{-iR}^{iR}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\Gamma(Re^{i\theta},t)Rie^{i\theta}\,d\theta dt -\\ &+\int_{-iR}^{iR}\int_{-iR}^{iR}\Gamma(s,t)\,dsdt\\ = J_1+J_2+J_3+J_4,&\text{ say.}\end{aligned}$$

Also I_1 equals the sum of residues with respect to t at those poles within $A'B'C'D'$ of the sum of residues with respect to s of $\Gamma(s,t)$ at its poles within $ABCA$. Now, under suitable convergence conditions,

$$J_2,\ J_3,\ J_1 \to 0 \quad \text{as} \quad R \to \infty.$$

Hence
$$I_1 \to \lim_{R\to\infty} J_4 \equiv I = -\tfrac{1}{4}\pi^{-2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \Gamma(s,t)\, ds dt.$$

But
$$I_1 \to \sum_{\mu=1}^{B} \sum_{\nu=1}^{D} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Gamma(b_\mu+m, d_\nu+n),$$

provided that these double series are absolutely convergent.

In the same way it follows from the integral

$$\int_{A'C'D'A'} \int_{ACDA} \Gamma(s,t)\, ds dt,$$

where $ACDA$ is a semicircular contour to the left of AC in the s-plane and $A'C'D'A'$ is a similar contour in the t-plane, that

$$I = \sum_{\mu=1}^{A} \sum_{\nu=1}^{C} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Gamma(-a_\mu-m, -c_\nu-n).$$

Hence, under convergence conditions

(i) $\mathrm{re}\{2+F+H+K-B-D+ + \sum (a+b+c+d-e-f-g-h-j-k)\} < 0,$

(ii) $\mathrm{re}\{1-B+F+K+\sum (a+b-e-f-j-k)\} < 0,$

(iii) $\mathrm{re}\{1-D+H+K+\sum (c+d-g-h-j-k)\} < 0,$

(iv) $A-B-E+F-J+K = 0,$

(v) $C-D-G+H-J+K = 0,$

we find that

$$-\tfrac{1}{4}\pi^{-2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \Gamma(s,t)\, ds dt$$

$$= \sum_{\mu=1}^{A} \sum_{\nu=1}^{C} \left\{ \Gamma(-a_\mu, -c_\nu) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{((b)+a_\mu)_m((d)+c_\nu)_n(1-(e)+a_\mu)_m}{(1+a_\mu-(a))_m(1+c_\nu-(c))_n((f)+a_\mu)_m} \times \right.$$
$$\left. \times \frac{(1-(g)+c_\nu)_n(1-(j)+a_\mu+c_\nu)_{m+n}(-1)^{(A+E+J)m+(C+G+J)n}}{((h)+c_\nu)_n((k)+a_\mu+c_\nu)_{m+n}} \right\}$$

$$= \sum_{\mu=1}^{B} \sum_{\nu=1}^{D} \left\{ \Gamma(b_\mu, d_\nu) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{((a)+b_\mu)_m((c)+d_\nu)_n(1-(f)+b_\mu)_m}{(1+b_\mu-(b))_m(1+d_\nu-(d))_n((e)+b_\mu)_m} \times \right.$$
$$\left. \times \frac{(1-(h)+d_\nu)_n(1-(k)+b_\mu+d_\nu)_{m+n}(-1)^{(B+F+K)m+(D+H+K)n}}{((g)+d_\nu)_n((j)+b_\mu+d_\nu)_{m+n}} \right\}. \quad (1)$$

Obviously this method can be extended to triple integrals and, in general, to multiple integrals, though the results become progressively more complicated.

2. A basic double integral

For the basic series, let

$$[a]_0 = 1, \qquad [a]_n = (1-q^a)(1-q^{a+1})\ldots(1-q^{a+n-1}),$$

and

$$\Pi[(a);(b)] = \prod_{n=1}^{\infty} \frac{(1-q^{a_1+n})\ldots(1-q^{a_A+n})}{(1-q^{b_1+n})\ldots(1-q^{b_B+n})}.$$

We suppose that q is real, $0 < q < 1$, and $q = \exp(-\tau)$, $\tau > 0$. This restriction is easily removed, for, when the result has been proved for q real, it follows at once for q complex, by analytic continuation over the circle $|q| < 1$. The restriction that q be real is introduced only to simplify the proof by ensuring that all the strip contours to be considered will lie parallel to the real axis. All the other parameters are assumed complex. Thus, (a), (b),... are sequences of complex numbers such that none of the members of the sequences

$$1-q^{-(a)-n}, \qquad 1-q^{(b)+n},$$

coincide for $n = 0, 1,\ldots$. Now

$$q^{a+s} = \exp\{-\tau(a+s)\}$$

and has a period of $2\pi i/\tau$. Hence $1/\Pi[a+s]$ has poles at all the points $s = -a-n+2\pi ik/\tau$ in the s-plane, one set of poles in every strip of width $2\pi/\tau$. Its residue at $s = -a-n$ is

$$(-1)^n q^{-\frac{1}{2}n(n+1)}/\tau[1]_n\,\Pi[1].$$

Let

$$\Pi(s,t) = \Pi\begin{bmatrix}(e)+s,(f)-s,(g)+t,(h)-t,(j)+s+t,(k)-s-t\\(a)+s,(b)-s,(c)+t,(d)-t\end{bmatrix}$$

and consider

$$-4\pi^2 I_1 = \int_{A'B'C'D'A'} \int_{ABCDA} \Pi(s,t)\,ds\,dt$$

taken round the contours $A(-i\pi/\tau)$, $B(-i\pi/\tau+R)$, $C(i\pi/\tau+R)$, $D(i\pi/\tau)$ and $A(-i\pi/\tau)$, $D(i\pi/\tau)$, $E(i\pi/\tau-R)$, $F(-i\pi/\tau-R)$ in the s-plane, and two similar contours $A'B'C'D'$ and $A'D'E'F'$ in the t-plane. The contours in the s-plane are indented to ensure that the first $R+1$ of each increasing sequence of poles fall within $ABCD$, and the first $R+1$ of each decreasing sequence of poles fall within $ADEF$. The contours $A'B'C'D'$ and $A'D'E'F'$ in the t-plane are similarly indented.

Then

$$-4\pi^2 I_1$$
$$= \int_{-i\pi/\tau}^{i\pi/\tau} \int_{-i\pi/\tau}^{i\pi/\tau} \{\Pi(R+s, R+t) - \Pi(s, R+t) - \Pi(R+s, t) + \Pi(s, t)\}\, ds dt$$
$$= J_1 - J_2 - J_3 + I, \text{ say.}$$

But I_1 is the sum of residues with respect to t at its poles in $A'B'C'D'A'$ of the sum of residues with respect to s of $\Pi(s, t)$ at its poles in $ABCDA$. Thus

$$I_1 \to \sum_{\mu=1}^{A} \sum_{\nu=1}^{C} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Pi(-a_\mu - m, -c_\nu - n) \quad \text{as } R \to \infty.$$

Also, under suitable convergence conditions,

$$J_1, J_2, J_3 \to 0 \quad \text{as } R \to \infty.$$

Hence
$$I_1 \to I \quad \text{as } R \to \infty.$$

Similarly, from

$$-4\pi^2 I_2 = \int_{A'D'E'F'A'} \int_{ADEFA} \Pi(s, t)\, ds dt$$

we have
$$-4\pi^2 I = \sum_{\mu=1}^{B} \sum_{\nu=1}^{D} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Pi(b_\mu + m, d_\nu + n).$$

The relations
$$\Pi[a+m] = \Pi[a]/[a]_m,$$

and
$$\Pi[a-m] = (-q^a)^m q^{-\frac{1}{2}m(m+1)} \Pi[a][1-a]_m,$$

are used to evaluate the series of residues, and we find, under the convergence conditions

I. $B+D > F+H+2K$
or $B+D = F+H+2K$ and $\operatorname{re} \sum (f+k+h-b-d) > 0$,

II. $B > F+K$
or $B = F+K$ and $\operatorname{re} \sum (f+k-b) > 0$,

III. $D > H+K$,
or $D = H+K$ and $\operatorname{re} \sum (h+k-d) > 0$,

IV. $A+C > E+G+2J$
or $A+C = E+G+2J$ and $\operatorname{re} \sum (e+g+j-a-c) > 0$,

V. $A > E+J$
or $A = E+J$ and $\operatorname{re} \sum (e+j-a) > 0$,

VI. $C > G+J$

or $C = G+J$ and $\mathrm{re}\sum(g+j-c) > 0$,

that

$$-\frac{1}{4\pi^2}\int_{-i\pi/\tau}^{i\pi/\tau}\int_{-i\pi/\tau}^{i\pi/\tau}\Pi(s,t)\,ds\,dt$$

$$=\sum_{\mu=1}^{A}\sum_{\nu=1}^{C}\left\{\Pi(-a'_\mu,-c'_\nu)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[(b)+a_\mu]_m[(d)+c_\nu]_n[1+a_\mu-(e)]_m}{[(f)+a_\mu]_m[(h)+c_\nu]_n[(k)+a_\mu+c_\nu]_{m+n}}\times\right.$$

$$\left.\times\frac{[1+c_\nu-(g)]_n[1+a_\mu+c_\nu-(j)]_{m+n}\,Q_1}{[1+a_\mu-(a)]_m[1+c_\nu-(c)]_n}\right\}$$

$$=\sum_{\mu=1}^{B}\sum_{\nu=1}^{D}\left\{\Pi(b'_\mu,d'_\nu)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[(a)+b_\mu]_m[(c)+d_\nu]_n[1+b_\mu-(f)]_m}{[(e)+b_\mu]_m[(g)+d_\nu]_n[(j)+b_\mu+d_\nu]_{m+n}}\times\right.$$

$$\left.\times\frac{[1+d_\nu-(h)]_n[1-d_\nu-b_\mu-(k)]_{m+n}\,Q_2}{[1+b_\mu-(b)]_m[1+d_\nu-(d)]_n}\right\}\qquad(2)$$

where

$$Q_1=(-1)^{(A-E-J)m+(C-G-J)n}\,q^{\Sigma(e+j-a)m+\Sigma(g+j-c)n}\times$$

$$\times q^{(A-E-J)a_\mu m+(C-G-J)c_\nu n-J(a_\mu m+c_\nu n)}\times$$

$$\times q^{(A-E)\frac{1}{2}m(m+1)+(C-G)\frac{1}{2}n(n+1)-\frac{1}{2}J(m+n)(m+n+1)},$$

and

$$Q_2=q^{(B-F)\frac{1}{2}m(m+1)+(D-H)\frac{1}{2}n(n+1)-\frac{1}{2}K(m+n)(m+n+1)}\times$$

$$\times(-1)^{(B-F-K)m+(D-H-K)n}q^{\Sigma(f+k-b)m+\Sigma(h+k-d)n}\times$$

$$\times q^{(B-F-K)b_\mu m+(D-H-K)d_\nu n-K(b_\mu m+d_\nu n)}.$$

The first set of double series are all convergent if

VII. $A > E+J$ or $C > G+J$ or $A = E$, $C = G$, $J = 0$,

and the second set of double series are all convergent if

VIII. $B > K+F$ or $D > K+H$ or $D = H$, $B = F$, $K = 0$.

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INVERSE FACTORIAL SERIES

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INVERSE factorial series are closely related to the interpolatory series treated in a previous paper (3). The following theorem will be proved.

If the coefficients in the inverse factorial series

$$\Gamma(z)A(z) = \frac{a_0}{z} + \frac{1!\,a_1}{z(z+1)} + \frac{2!\,a_2}{z(z+1)(z+2)} + \dots \tag{1}$$

satisfy the condition

$$\sum_{m=0}^{\infty} 2^m |a_m| < \infty, \tag{2}$$

then the series converges for all finite z and

$$A(z) = \frac{1}{\Gamma(z)} \sum_{r=0}^{\infty} \frac{(-1)^r A(-r)}{r!\,(z+r)}. \tag{3}$$

Convergence of $A(z)$ for all finite z follows immediately from Landau's expression (1) for the abscissa of convergence

$$\lim \log \Big| \sum_{m=n}^{\infty} a_m \Big| \Big/ \log n$$

since

$$\Big| \sum_{m=n}^{\infty} a_m \Big| \leqslant \frac{1}{2^n} \sum_{m=n}^{\infty} |a_m| 2^m < \frac{1}{2^n}$$

for n sufficiently great because of (2).

To prove (3) we start from the elementary identity

$$\frac{a_0}{z} + \frac{1!\,a_1}{z(z+1)} + \dots + \frac{n!\,a_n}{z(z+1)\dots(z+n)} = \sum_{r=0}^{n} \frac{(-1)^r b_r^{(n)}}{r!\,(z+r)}, \tag{4}$$

where

$$b_r^{(n)} = r!\,a_r + \frac{(r+1)!\,a_{r+1}}{1!} + \dots + \frac{n!\,a_n}{(n-r)!}. \tag{5}$$

I show that the double series

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(z+r)} \sum_{p=r}^{\infty} \frac{p!\,a_p}{(p-r)!}$$

converges absolutely and so justify the equation

$$\frac{a_0}{z} + \frac{1!\,a_1}{z(z+1)} + \dots = \sum_{r=0}^{\infty} \frac{(-1)^r b_r}{r!\,(z+r)}, \tag{6}$$

where

$$b_r = \sum_{p=r}^{\infty} \frac{p!\,a_p}{(p-r)!}. \tag{7}$$

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If $z \neq 0, -1, -2, \ldots$, we have

$$\sum_{r=0}^{\infty} \frac{1}{r!\,|z+r|} \sum_{p=r}^{\infty} \frac{p!\,|a_p|}{(p-r)!} < K \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{p=r}^{\infty} \frac{p!\,|a_p|}{(p-r)!}$$

$$= K \sum_{p=0}^{\infty} |a_p| \sum_{r=0}^{p} \frac{p!}{r!\,(p-r)!} = K \sum_{p=0}^{\infty} |a_p| 2^p < \infty,$$

from (2), which proves the assertion. To obtain (3) we note that

$$b_r = \sum_{p=r}^{\infty} \frac{\Gamma(p+1)a_p}{\Gamma(p-r+1)} = \sum_{p=0}^{\infty} \frac{\Gamma(p+1)a_p}{\Gamma(p-r+1)} = A(-r) \tag{8}$$

and multiply (6) by $1/\Gamma(z)$.

The convergence of the double series

$$\sum_{r=0}^{\infty} \frac{1}{r!} \sum_{p=r}^{\infty} \frac{p!\,|a_p|}{(p-r)!}$$

implies that

$$\sum_{r=0}^{\infty} \left|\frac{b_r}{r!}\right| < \infty,$$

and therefore

$$b_r = o(r!), \tag{9'}$$

$$\sum_{r=1}^{\infty} \frac{1}{r} \left|\frac{b_r}{r!}\right| < \infty \tag{9''}$$

if b_r is defined by (7).

Naturally (3) holds under much less restrictive conditions than (2). A well-known necessary and sufficient condition (**2**) that a function $f(z) = \Gamma(z)A(z)$ can be represented by a convergent factorial series

$$f(z) = \sum_{n=0}^{\infty} \frac{n!\,a_n}{z(z+1)\ldots(z+n)}$$

is that it be a Laplace integral

$$f(z) = \int_0^1 u^{z-1}\phi(u)\,du \tag{10}$$

with

$$\phi(u) = \sum_{n=0}^{\infty} a_n(1-u)^n \tag{11}$$

and

$$a_n = O(n^k)$$

for some positive k. Obviously the series (11) converges for $|1-u| < 1$.

If we suppose (11) to converge for $|1-u| < 1+\delta$, $\delta > 0$, then $\phi(u)$ can be expanded in the form

$$\phi(u) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} b_r u^r \tag{12}$$

with b_r given by (8). If the series (12) is such that

$$\sum_{r=0}^{\infty} \frac{(-1)^r b_r}{r!\,(z+r)}$$

converges, (10) can be integrated term by term for $\operatorname{re} z > 0$ and we again obtain (6).

In (3) the following consistency property† is proved: *if*

$$A(z) = \frac{1}{\Gamma(z)} \sum_{r=0}^{\infty} \frac{(-1)^r b_r}{r!\,(z+r)}, \tag{13}$$

where
$$\sum_{r=1}^{\infty} \frac{1}{r} \left|\frac{b_r}{r!}\right| < \infty, \qquad \frac{b_r}{r!} = o(r^{\delta}), \quad \delta > 0 \text{ arbitrary}, \tag{14}$$

then
$$A(z) = \frac{1}{\Gamma(z-\lambda)} \sum_{n=0}^{\infty} \frac{(-1)^n A(-n+\lambda)}{n!\,(z+n-\lambda)} \tag{15}$$

for $\operatorname{re}\lambda > 0$.

This can be translated into a property of factorial series if (3) is equivalent to (1), e.g. if (2) holds. Incidentally (2) implies (9′), (9″), conditions sufficient for (14) to hold.

Examples: (i) $a_0 = 1$, $a_m = 0$ $(m \geqslant 1)$.

The consistency relation (15) gives

$$\frac{1}{\Gamma(w+1)} = \frac{1}{\Gamma(w-\lambda)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1-n+\lambda)(w+n-\lambda)}$$

for $\operatorname{re}\lambda > 0$. Substituting $w = z+a-1$, $\lambda = a-1$, we get

$$\frac{1}{\Gamma(z+a)} = \frac{1}{\Gamma(z)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(a-n)(z+n)},$$

i.e.
$$\frac{\Gamma(z)\Gamma(a+1)}{\Gamma(z+a)} = \sum_{n=0}^{\infty} \frac{(-1)^n a(a-1)\ldots(a-n)}{n!\,(z+n)}$$

for $\operatorname{re} a > 1$. Analytic continuation shows that this well-known result (4) holds for $\operatorname{re} a > 0$.

(ii)
$$(\tfrac{1}{2}x)^{-\nu} J_{\nu}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (\tfrac{1}{2}x)^{2m}}{m!\,\Gamma(m+\nu+1)}.$$

From (3),
$$(\tfrac{1}{2}x)^{-\nu} J_{\nu}(x) = \frac{1}{\Gamma(\nu)} \sum_{r=0}^{\infty} \frac{(\tfrac{1}{2}x)^r J_r(x)}{r!\,(\nu+r)}$$

† There is a minor change of notation: we replace z by $-z$ and λ by $-\lambda$.

and, from (15),

$$(\tfrac{1}{2}x)^{-\nu}J_\nu(x) = \frac{1}{\Gamma(\nu-\lambda)}\sum_{n=0}^{\infty}\frac{(-1)^n(\tfrac{1}{2}x)^{n-\lambda}J_{-n+\lambda}(x)}{n!\,(n+\nu-\lambda)}.$$

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A TWO-POINT BOUNDARY PROBLEM FOR INFINITELY DIFFERENTIABLE FUNCTIONS

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1. MANY years ago I proposed the following problem [(**4**) 465]:

Let $(p; q)$ be any complete pair of sequences and $f(z)$, $g(z)$ any integral functions. Is it always possible to find an integral function $h(z)$ such that

$$h^{(p_n)}(1) = f^{(p_n)}(1), \qquad h^{(q_n)}(0) = g^{(q_n)}(0) \quad (n \geqslant 1)? \tag{1}$$

A pair of sequences $(p_1, p_2,\ldots; q_1, q_2,\ldots)$ is said to be *complete* if there are, so to speak, just enough equations (1) to determine the unknown function $h(z)$ in favourable cases: to be precise, if the operators $f^{(p_n)}(1)$, $f^{(q_n)}(0)$ form a basic set. Denoting by $D(m)$ the number of p's and q's which are less than m, the necessary and sufficient conditions for a complete pair are

$$D(m) \geqslant m \quad (m \geqslant 1) \tag{2}$$

and

$$D(m_r) = m_r, \quad \text{for an infinite sequence } m_1, m_2,\ldots. \tag{3}$$

It was shown that the problem can be answered affirmatively in the case of the pair $(0, 2, 4,\ldots; 0, 2, 4,\ldots)$. The resulting theorem is equivalent† to the fundamental theorem of Guichard's that, *if $f(z)$ is any integral function, an integral function $g(z)$ can be found such that*

$$g(z+1)-g(z) = f(z). \tag{4}$$

The case of the pair $(0, 2, 4,\ldots; 1, 3, 5,\ldots)$ was likewise answered affirmatively. In both these cases the pairs have an obviously cyclic character and Pólya (**1**) has shown that the results can be extended to a much wider class of pairs, likewise possessing a cyclic character. More recently Vermes (**3**) has introduced the valuable notion of a *complete pair of bounded span*, i.e. one for which the inequality

$$m_{r+1}-m_r \leqslant K \tag{5}$$

holds in addition to (2), (3), and has shown that the equations (1) possess a unique solution in certain cases. But no substantial progress has yet been made for a general complete pair.

My object here is to show that the problem, and indeed much more general problems of the same kind, can be solved if, instead of dealing with integral functions, we consider functions of a real variable possessing

† Proof partly in (**4**), 457 and partly in § 5 below.

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differential coefficients of all orders in $-\infty < x < \infty$. In the subsequent work all numbers will be assumed to be real. I prove

THEOREM 1. *Given any complete pair* $(p; q)$ *and any sequences* $\{A_n\}, \{B_n\}$, *a function* $f(x)$ *possessing differential coefficients of all orders in*

$$-\infty < x < \infty$$

can be found such that

$$f^{(p_n)}(1) = A_n, \qquad f^{(q_n)}(0) = B_n \quad (n \geqslant 1). \tag{6}$$

2. LEMMA 1. *Given an integer* p, *a positive number* δ, *and two sets of numbers* $a_1, a_2, \ldots, a_n$ *and* $A_1, A_2, \ldots, A_n$, *there is a trigonometric polynomial* $F(x)$ *such that*

$$|F^{(k)}(x)| < \delta \quad (-\infty < x < \infty;\ k = 0, 1, \ldots, p-1), \tag{7}$$

and

$$|F^{(p)}(a_1) - A_1| + \ldots + |F^{(p)}(a_n) - A_n| < \delta. \tag{8}$$

Suppose first that $0 < a_1 < a_2 < \ldots < a_n < 2\pi$ and take

$$F(x) = \sum_{m=1}^{\nu} d_m \cos(mx - \theta_m),$$

where $d_m = \pm D/m^{p+1}$, D being a constant to be chosen later, and the signs and phases are chosen as follows.

Let α be the first integer such that

$$|d_1 + 2^p d_2 + \ldots + \alpha^p d_\alpha - A_1| \leqslant D,$$

the d's being taken positive if $A_1 \geqslant 0$ and negative if $A_1 < 0$. Moreover, take

$$\theta_m = ma_1 + \tfrac{1}{2}p\pi \quad (m = 1, 2, \ldots, \alpha).$$

Next, let β be the first integer such that

$$|(\alpha+1)^p d_{\alpha+1} + \ldots + \beta^p d_\beta - A_2| \leqslant D,$$

the new d's having the sign of A_2, and let

$$\theta_m = ma_2 + \tfrac{1}{2}p\pi \quad (m = \alpha+1, \alpha+2, \ldots, \beta);$$

$\gamma, \delta, \ldots, \nu$ are defined similarly. Then

$$\begin{aligned}
|F^{(p)}(a_1) - A_1| &= \Big| \sum_{m=1}^{\alpha} d_m m^\alpha + \sum_{m=\alpha+1}^{\beta} d_m m^p \cos m(a_1 - a_2) + \ldots + \\
&\qquad + \sum_{m=\mu+1}^{\nu} d_m m^p \cos m(a_1 - a_n) - A_1 \Big| \\
&\leqslant D + D\Big| \sum_{m=\alpha+1}^{\beta} m^{-1} \cos m(a_1 - a_2) \Big| + \ldots + D\Big| \sum_{m=\mu+1}^{\nu} m^{-1} \cos m(a_1 - a_n) \Big| \\
&\leqslant D\left\{1 + \frac{1}{\alpha+1} \operatorname{cosec} \tfrac{1}{2}(a_2 - a_1) + \ldots + \frac{1}{\mu+1} \operatorname{cosec} \tfrac{1}{2}(a_n - a_1)\right\} \\
&\leqslant nD \max\{1, \operatorname{cosec} \tfrac{1}{2}(a_2 - a_1), \operatorname{cosec} \tfrac{1}{2}(a_3 - a_2), \ldots, \operatorname{cosec} \tfrac{1}{2}(a_n - a_{n-1})\} \\
&= nDM \text{ (say).}
\end{aligned}$$

The same inequality holds in the case of the other a's. Moreover

$$|F^{(k)}(x)| \leqslant \sum_{m=1}^{N} |d_m| m^k < D \sum_{m=1}^{\infty} m^{-2} \quad (k = 0, 1, \ldots, p-1),$$

so that it is sufficient to take $D = \delta/2n^2M$.

If $a_1, a_2, \ldots$ do not lie in $(0, 2\pi)$, let A, B be such that

$$a_1' = Aa_1 + B, \quad a_2' = Aa_2 + B, \quad \ldots$$

all lie in $(0, 2\pi)$. As above, find a trigonometric polynomial $G(x)$ such that

$$|G^{(k)}(x)| \leqslant A^{-k}\delta \quad (k = 0, 1, \ldots, p-1),$$

$$|G^{(p)}(a_1') - A_1 A^{-p}| + \ldots + |G^{(p)}(a_n') - A_n A^{-p}| < A^{-p}\delta.$$

$F(x) = G(Ax + B)$ is then the function required.

LEMMA 2. *Given*

(i) *a positive decreasing sequence* $\{\delta_p\}$;
(ii) *an increasing sequence of integers* $\{\nu(p)\}$;
(iii) *a sequence* $\{a_n\}$;
(iv) *a double sequence* $\{A_{pn}\}$ $(n = 1, 2, \ldots, \nu(p);\ p \geqslant 1)$;

then there is a function $F(x)$ *possessing differential coefficients of all orders in* $-\infty < x < \infty$ *such that*

$$|F(x)| < \delta_0 \quad (-\infty < x < \infty) \tag{9}$$

and

$$|F^{(p)}(a_1) - A_{p1}| + \ldots + |F^{(p)}(a_{\nu(p)}) - A_{p\nu(p)}| < \delta_p \quad (p \geqslant 1). \tag{10}$$

In accordance with Lemma 1, there is a trigonometric polynomial $F_1(x)$ such that

$$|F_1(x)| < \tfrac{1}{2}\delta_0,$$

$$|F_1'(a_1) - A_{11}| + \ldots + |F_1'(a_{\nu(1)}) - A_{1\nu(1)}| < \tfrac{1}{2}\delta_1,$$

and a similar polynomial $F_2(x)$ with the properties

$$|F_2(x)| < \frac{1}{2^2}\delta_0, \qquad |F_2'(x)| < \frac{1}{2^2}\delta_1/\nu(1),$$

$$|F_2''(a_1) - \{A_{21} - F_1''(a_1)\}| + \ldots + |F_2''(a_{\nu(2)}) - \{A_{2\nu(2)} - F_1''(a_{\nu(2)})\}| < \tfrac{1}{2}\delta_2,$$

and so on; e.g. the properties of $F_3(x)$ are

$$|F_3(x)| < \frac{1}{2^3}\delta_0, \qquad |F_3'(x)| < \frac{1}{2^3}\delta_1/\nu(1), \qquad |F_3''(x)| < \frac{1}{2^3}\delta_2/\nu(2),$$

$$|F_3'''(a_1) - \{A_{31} - F_1'''(a_1) - F_2'''(a_1)\}| + \ldots < \tfrac{1}{2}\delta_3.$$

It is evident that the series

$$F(x) = F_1(x) + F_2(x) + \ldots$$

can be differentiated any number of times and that

$$|F^{(p)}(a_1)-A_{p1}|+\ldots+|F^{(p)}(a_{\nu(p)})-A_{p\nu(p)}|$$
$$\leqslant |F_1^{(p)}(a_1)+\ldots+F_p^{(p)}(a_1)-A_{p1}|+\sum_{q=p+1}^{\infty}|F_q^{(p)}(a_1)|+\ldots$$
$$<\tfrac{1}{2}\delta_p+\nu(p)\sum_{q=p+1}^{\infty}\frac{1}{2^q}\delta_p/\nu(p)<\delta_p.$$

3. Lemma 2 can be modified in various ways. Thus, let $f(x)$ be a function continuous in a finite interval (a,b). Then (9) can be replaced by

$$|F(x)-f(x)|<\delta_0 \quad (a\leqslant x\leqslant b). \tag{11}$$

For, by the fundamental theorem of Weierstrass, there is a polynomial $P(x)$ such that

$$|f(x)-P(x)|<\tfrac{1}{2}\delta_0 \quad (a\leqslant x\leqslant b).$$

By Lemma 2, there is a function $G(x)$ such that

$$|G(x)|<\tfrac{1}{2}\delta_0 \quad (-\infty<x<\infty)$$

and
$$|G^{(p)}(a_1)-\{A_{p_1}-P^{(p)}(a_1)\}|+\ldots<\delta_p \quad (p\geqslant 1).$$

Thus $F(x)=G(x)+P(x)$ satisfies the required conditions. Moreover it is evident from this construction that a trigonometric polynomial $F(x)$ can be found to satisfy (11) together with (10) for $p=1, 2,\ldots, q$. One consequence of this is sufficiently curious to deserve mention.

THEOREM 2. *Given*

(i) *a function $f(x)$ continuous in (a,b);*
(ii) *a sequence $\{a_n\}$ $(n\geqslant 1)$;*
(iii) *a double sequence $\{A_{pn}\}$ $(p\geqslant 1,\ n\geqslant 1)$;*

there is a sequence $\{F_k(x)\}$ of trigonometric polynomials such that

$$F_k(x)\to f(x) \text{ as } k\to\infty, \text{ uniformly in } (a,b)$$

and
$$F_k^{(p)}(a_n)\to A_{pn} \text{ as } k\to\infty, \text{ for all } p, n.$$

$F_k(x)$ is the trigonometric polynomial such that

$$|F_k(x)-f(x)|<k^{-1} \quad (a\leqslant x\leqslant b)$$

and
$$|F_k^{(p)}(a_1)-A_{p1}|+\ldots+|F_k^{(p)}(a_k)-A_{pk}|<k^{-1} \quad (p=1, 2,\ldots, k),$$

so that
$$|F_k^{(p)}(a_n)-A_{pn}|<k^{-1} \quad (k\geqslant \max(p,n)).$$

4. Coming now to the proof of Theorem 1, we observe first [(**5**), 50] that the equations (6) are satisfied by a unique integral function $f(x)$, provided that A_n, B_n tend to zero sufficiently rapidly, say if

$$|A_n|+|B_n|<\psi(n).$$

In the general case, construct, in accordance with Lemma 2, a function $F(x)$ such that

$$|F^{(p_n)}(1)-A_n|+|F^{(q_n)}(0)-B_n| < \psi(n) \quad (n \geqslant 1),$$

and write $A'_n = A_n - F^{(p_n)}(1), \qquad B'_n = B_n - F^{(q_n)}(0).$

Let $G(x)$ be the solution of the equations

$$G^{(p_n)}(1) = A'_n, \qquad G^{(q_n)}(0) = B'_n.$$

Then $f(x) = F(x)+G(x)$ is the required solution.

The same method can be used in many similar problems, e.g.

THEOREM 3. *Let $\{a_n\}$, $\{A_n\}$ be any sequences. Then there is a function $f(x)$, possessing differential coefficients of all orders in $-\infty < x < \infty$, such that*

$$f^{(n)}(a_n) = A_n \quad (n \geqslant 0).$$

5. Two problems arise in connexion with Theorem 1. Bearing in mind the connexion between Guichard's theorem and the two-point problem for integral functions, is it true that, if $f(x)$ is a given function possessing differential coefficients of all orders in $-\infty < x < \infty$, then there is a function $g(x)$ of the same kind such that

$$g(x+1)-g(x) = f(x)?$$

In the case of integral functions the proof is as follows. Given $f(z)$, write $f(z) = f_1(z)+f_2(z)$, where $f_1(z)$ is even and $f_2(z)$ odd. Find $h_1(z)$ so that

$$h_1^{(2n)}(\tfrac{1}{2}) = \tfrac{1}{2}f_1^{(2n)}(0), \qquad h_1^{(2n)}(0) = 0 \qquad (n \geqslant 0).$$

Then $h_1(z)$ is an odd function and

$$h_1(z+\tfrac{1}{2})-h_1(z-\tfrac{1}{2}) = \sum_{n=0}^{\infty} \frac{z^n}{n!}\{h_1^{(n)}(\tfrac{1}{2})-h_1^{(n)}(-\tfrac{1}{2})\}$$

$$= 2\sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} h_1^{(2n)}(\tfrac{1}{2}) = f_1(z).$$

Similarly find $h_2(z)$ so that

$$h_2^{(2n+1)}(\tfrac{1}{2}) = \tfrac{1}{2}f_2^{(2n+1)}(0), \qquad h_2^{(2n+1)}(0) = 0,$$

and, in consequence,

$$h_2(z+\tfrac{1}{2})-h_2(z-\tfrac{1}{2}) = f_2(z).$$

Then $g(z) = h_1(z-\tfrac{1}{2})+h_2(z-\tfrac{1}{2})$ is the required solution. However, this argument does not extend to infinitely differentiable functions, since there is no longer a Taylor expansion.

6. The second problem is also suggested by known properties of integral functions. When these occur in the data of interpolatory problems, it is generally possible to relate the order and type of the

solution to those of the given functions. For example, in Guichard's theorem it is known that $g(z)$ can be found of the same order [(**5**), 22] and type (**2**) as $f(z)$. Infinitely differentiable functions can be classified to a certain extent by the rate of growth of the derivatives and, if this is not too rapid, by quasi-analytic classes. It is natural to inquire if, in results such as Theorems 1 and 3, it is possible to relate the rate of growth of the solution to that of the given sequences. However, the method of approximation used in the proof of Theorem 1 is not well adapted to deal with questions of this type.

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THE COEFFICIENT THEORY OF INTEGRAL FUNCTIONS WITH DOMINANT EXPONENTIAL PARTS

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1. Introduction

We are concerned here mainly with integral functions of order 1 and type h_1 (> 0),

$$F(z) = \sum_0^\infty a_n z^n \equiv \sum_0^\infty \frac{c_n}{n!} z^n. \tag{1}$$

The Laplace transform of such a function [(**3**) § 1, (**5**) §§ 20, 21],

$$f(z) = \sum_0^\infty n!\, a_n z^{-n-1} \equiv \sum_0^\infty c_n z^{-n-1}, \tag{2}$$

is then regular outside the circle $|z| = h_1$ and singular at one point on the circumference of this circle at least.

If $f(z)$ is meromorphic on the circle, it can be written in the form

$$f(z) = \sum_{\nu=1}^{q} \left\{ \sum_{\mu=1}^{m_\nu} \frac{A_{\mu\nu}}{(z-\alpha_\nu)^\mu} \right\} + f^*(z), \tag{3}$$

where each $|\alpha_\nu| = h_1$ and $f^*(z)$ is regular outside $|z| = h^* < h_1$.

The inverse transform of (3) takes the form

$$F(z) = \sum_{\nu=1}^{q} \left(\sum_{\mu=1}^{m_\nu} A_{\mu\nu} z^{\mu-1} \right) e^{\alpha_\nu z} + F^*(z), \tag{4}$$

where $F^*(z)$ is an integral function of order at most 1 and type $h^* < h_1$. It may happen, however, that $f(z)$ is meromorphic on a succession of circles $|z| = h_1$, $|z| = h_2$,... with $h_1 > h_2 ... > h^*$ and in this case (3) and (4) continue to hold with the proviso that $|\alpha_1| \geqslant |\alpha_2| \geqslant ... \geqslant |\alpha_\nu| ... > h^*$.

The dominant portion of (4) consists in these cases of exponentials of type h_1 multiplied by appropriate polynomials in z, which may happen to be constants. Again, there may be secondary sets of exponentials of type $h_2 < h_1$, tertiary sets of type $h_3 < h_2$, and so on.

I first obtain necessary and sufficient conditions in terms of the coefficients a_n of $F(z)$ of (1) that this may be so. I then determine the values of the constants α_ν, $A_{\mu\nu}$ arising in (4) in terms of the coefficients a_n of (1). The method used is to apply the classical Hadamard theory [(**2**) §§ 14–22, (**1**) § 85, (**4**) § 6] to $f(z)$ of (2) and then employ the relations between c_n and a_n to get the constants α_ν and $A_{\mu\nu}$ in terms of the a_n. Hadamard's classification of functions meromorphic in a circle [(**2**) § 22,

(1) 335] is then applied to get a natural classification of integral functions with dominant exponential parts. Finally, some extensions of the theory to integral functions of any finite order and mean type are given.

2. Functions analytic near infinity

The coefficient theory of functions regular at the origin but meromorphic in its neighbourhood was developed by Hadamard in his thesis. For our purpose it is necessary to restate this theory in terms of functions $f(z)$ of (2) which are zero at infinity and regular there but meromorphic in its neighbourhood. The relevant results are stated in the theorem below, in which I write $D_{n,s}$ for the Hankel determinant

$$|c_{n+\mu+\nu}| \quad (\mu, \nu = 0, 1, 2,..., s). \tag{5}$$

THEOREM 1. *Necessary and sufficient conditions that $f(z)$ should have on its circle of convergence $|z| = h_1$ exactly p_1 poles and no other singularities; on its next circle of meromorphy $|z| = h_2$ exactly p_2 poles, and so on; are that*

$$\limsup_{n\to\infty} |D_{n,\nu}|^{1/n} = h_1^{\nu+1} \quad (\nu = 0, 1, 2,..., p_1-2),$$

$$\lim_{n\to\infty} |D_{n,p_1-1}|^{1/n} = h_1^{p_1},$$

$$\limsup_{n\to\infty} |D_{n,p_1+\nu}|^{1/n} = h_1^{p_1} h_2^{\nu+1} \quad (\nu = 0, 1, 2,..., p_2-2),$$

$$\lim_{n\to\infty} |D_{n,p_1+p_2-1}|^{1/n} = h_1^{p_1} h_2^{p_2},$$

and so on. The affixes of the p_1 poles are given by the zeros of the polynomial

$$\lim_{n\to\infty} K_{n,p_1}(z)/D_{n,p_1-1}, \tag{6}$$

where

$$K_{n,\nu}(z) \equiv \begin{vmatrix} 1 & z & \cdot & \cdot & \cdot & z^{\nu} \\ c_n & c_{n+1} & \cdot & \cdot & \cdot & c_{n+\nu} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{n+\nu-1} & c_{n+\nu} & \cdot & \cdot & \cdot & c_{n+2\nu-1} \end{vmatrix}$$

and the affixes of the p_2 poles are given by the additional zeros of the polynomial

$$\lim_{n\to\infty} K_{n,p_1+p_2}(z)/D_{n,p_1+p_2-1}, \tag{7}$$

and so on. The zeros are counted according to multiplicity and the poles are of the same multiplicities as the respective zeros α_ν.

Here the polynomials (6), (7), and any subsequent ones, determine the constants α_ν and the degrees m_ν of the polynomials in (4). To get the equations for the $A_{\mu\nu}$ we have to develop (3) in powers of z^{-1} and identify the coefficients of the corresponding powers with those in (2). Thus

$$c_{n+t} = \sum_{\nu=1}^{q} \Big(\sum_{\mu=1}^{m_\nu} {}^{-\mu}C_{n+t-\mu+1}\, A_{\mu\nu}\, \alpha^{n+t-\mu+1} \Big) + O(h^{*n}) \quad (t = 1, 2,..., p), \tag{8}$$

where p is the total number of poles under consideration, counted according to multiplicity. Equations (8) give, with appropriate determinations of sign,

$$A_{\mu\nu} = \lim_{n\to\infty} \Delta/M_{11}, \tag{9}$$

Δ denoting

$$\begin{vmatrix} 0 & 0 & . & 1 & . & 0 \\ c_{n+1} & {}^{-1}C_{n+1}\alpha_1^{n+1} & . & {}^{-\mu}C_{n+2-\mu}\alpha_\nu^{n+2-\mu} & . & {}^{-m_q}C_{n+2-m_q}\alpha_q^{n+2-m_q} \\ . & . & . & . & . & . \\ c_{n+p} & {}^{-1}C_{n+p}\alpha_1^{n+p} & . & {}^{-\mu}C_{n+p-\mu+1}\alpha_\nu^{n+p-\mu+1} & . & {}^{-m_q}C_{n+p-m_q+1}\alpha_q^{n+p-m_q+1} \end{vmatrix}$$

and M_{11} the minor of the top left-hand element of the determinant Δ.

In particular, if the poles are all simple, then $q = p$ and, to within sign, writing A_ν for $A_{1\nu}$,

$$A_\nu = \lim_{n\to\infty} \begin{vmatrix} 0 & 0 & . & 1 & . & 0 \\ c_{n+1} & \alpha_1^{n+1} & . & \alpha_\nu^{n+1} & . & \alpha_p^{n+1} \\ . & . & . & . & . & . \\ c_{n+p} & \alpha_1^{n+p} & . & \alpha_\nu^{n+p} & . & \alpha_p^{n+p} \end{vmatrix} \Big/ \prod_1^p \alpha_\mu^{n+1} \times \prod_{\lambda<\mu}^p (\alpha_\lambda - \alpha_\mu), \tag{10}$$

i.e.

$$A_\nu = \lim_{n\to\infty} \frac{b_0 c_{n+1} + b_1 c_{n+2} + \ldots + b_{p-1} c_{n+p}}{\alpha_\nu^{p+1} P_\nu(z)},$$

where

$$P_\nu(z) \equiv \prod_{\mu=1}^{p}{}' (z-\alpha_\mu) \equiv \sum_0^{p-1} b_s z^s,$$

the factor $(z-\alpha_\nu)$ being missing from the product.

3. Coefficient theory

Bearing in mind the connexion between (3) and (4), and making use of the relation $c_n = n!\,a_n$, we can restate Theorem 1 as a theorem concerning integral functions with dominant exponential parts. In doing so I shall write $\mathfrak{D}_{n,s}$ for the determinant

$$|(n+\mu+\nu)!\,a_{n+\mu+\nu}| \quad (\mu, \nu = 0, 1, 2, \ldots, s)$$

deduced from (5), and $\mathfrak{K}_{n,\nu}(z)$ for the expression obtained by replacing in $K_{n,\nu}(z)$ of (6) every $c_{n+\nu}$ by $(n+\nu)!\,a_{n+\nu}$. We thus obtain

THEOREM 2. *Necessary and sufficient conditions that an integral function* $\sum a_n z^n$ *of order* 1 *and mean type shall have dominant exponential parts of type* h_1*; secondary exponential parts of type* h_2*, and so on; are that*

$$\limsup_{n\to\infty} |\mathfrak{D}_{n,\nu}|^{1/n} = h_1^{\nu+1} \quad (\nu = 0, 1, 2, \ldots, p_1-2), \qquad \lim_{n\to\infty} |\mathfrak{D}_{n,p_1-1}|^{1/n} = h_1^{p_1},$$

$$\limsup_{n\to\infty} |\mathfrak{D}_{n,p_1+\nu}|^{1/n} = h_1^{p_1} h_2^{\nu+1} \quad (\nu = 0, 1, 2, \ldots, p_2-2),$$

$$\lim_{n\to\infty} |\mathfrak{D}_{n,p_1+p_2-1}|^{1/n} = h_1^{p_1} h_2^{p_2},$$

and so on. The exponents of the elements of type h_1 are given by the zeros of the polynomial

$$\lim_{n\to\infty} \mathfrak{R}_{n,p_1}(z)/\mathfrak{D}_{n,p_1-1},$$

the exponents of the elements of type h_2 are given by the additional zeros of the polynomial

$$\lim_{n\to\infty} \mathfrak{R}_{n,p_1+p_2}(z)/\mathfrak{D}_{n,p_1+p_2-1},$$

and so on. The orders of the polynomials occurring in (4) *are determined by the multiplicities of the corresponding zeros α_ν in the above polynomials.*

The constants $A_{\mu\nu}$ arising in the polynomials of (4) are given by (8) and (9) or (10), as the case may be, after replacing each $c_{n+\nu}$ by $(n+\nu)!\,a_{n+\nu}$ $(\nu = 1, 2, \ldots, p)$.

4. Classification

In accordance with Hadamard's classification (**2**) the following cases may arise in functions having the form of $f(z)$ of (3):

(i) $f(z)$ is rational; (ii) $f(z)$ has only a finite number of poles together with an essential point at the origin; (iii) $f(z)$ has an essential point at the origin which is a limit-point of poles; (iv) $f(z)$ has on its circle of meromorphy at least one singularity which is non-polar, but outside it only a finite number of poles; (v) $f(z)$ has on its circle of meromorphy an essential point which is a limit-point of poles, but no other singularity of this kind.

In cases (i), (ii), (iv), $f(z)$ takes the form (3) in which now the affixes of the poles α_ν are not necessarily all on the circle of convergence. They all lie outside the circle of meromorphy and are distributed between this circle and the circle of convergence, there being at least one pole on the circumference of the latter. In case (i), $f^*(z)$ is a polynomial in z^{-1} which may reduce to a constant or be zero; in case (ii), $f^*(z)$ has an isolated essential point at the origin and no other singularities; in case (iv), $f^*(z)$ is regular outside the circle of meromorphy of $f(z)$.

In case (iii), $f(z)$ takes the form, in which $\lim \alpha_n = 0$ $(n \to \infty)$,

$$f(z) = \sum_{\nu=1}^{\infty} \left\{ \sum_{\mu=1}^{m_\nu} \frac{A_{\mu\nu}}{(z-\alpha_\nu)^\mu} - Q_\nu(z^{-1}) \right\} + f^*(z), \tag{11}$$

where the $Q_\nu(z^{-1})$ are polynomials introduced to secure uniform convergence of the Mittag–Leffler expansion [(**1**) § 76] outside the neighbourhood of the poles and the origin and $f^*(z)$ is the expansion of an isolated essential point at the origin.

In case (iv), $f(z)$ takes the form, in which $\lim \alpha_n = \alpha$, a point on the circle of meromorphy,

$$f(z) = \sum_{\nu=1}^{\infty} \left\{ \sum_{\mu=1}^{m_\nu} \frac{A_{\mu\nu}}{(z-\alpha_\nu)^\mu} - Q_\nu\left(\frac{1}{z-\alpha}\right) \right\} + f^*(z), \tag{12}$$

where the $Q_\nu\{(z-\alpha)^{-1}\}$ are polynomials introduced to secure uniform convergence of the Mittag–Leffler expansion outside the circle of meromorphy $|z| = \alpha$ except at the poles α_ν, and $f^*(z)$ is regular outside the circle $|z| = \alpha$. It will be noted that $f(z)$ may have on $|z| = \alpha$ other singularities than that at $z = \alpha$.

The operation of the inverse Laplace transform presents no difficulties in the first three cases. We get, in fact, formula (4) where, in case (i), $F^*(z)$ is a polynomial which may reduce to a constant or be zero; in case (ii), $F^*(z)$ is an integral function of order at most 1 and minimum type; in case (iv), $F^*(z)$ is an integral function of order at most 1 and type h^*: that is, type less than every $|\alpha_\nu|$.

Applying the inverse transform process to (11) we get, with $\lim \alpha_n = 0$,

$$F(z) = \sum_{\nu=1}^{\infty} \left\{ \left(\sum_{\mu=1}^{m_\nu} A_{\mu\nu} z^{\mu-1} \right) e^{\alpha_\nu z} - S_\nu(z) \right\} + F^*(z). \tag{13}$$

Here the polynomial $S_\nu(z)$ removes the same number of initial terms from the expansion in the brackets $\{\}$ as $Q_\nu(z^{-1})$ does from the expansion in terms of z^{-1} in the corresponding bracket in (11).

Let $g(z)$ consist of all those terms of (11) for which $|\alpha_\nu| < \theta r$, where $0 < \theta < 1$. By an appropriate determination of each corresponding $Q_\nu(z^{-1})$, as in Mittag–Leffler's proof (**1**), we ensure that, for $|z| > r$, $|g(z)| < \epsilon$, given. Then $G(z)$, the corresponding 'remainder' function in the inverse transform, is given by

$$G(z) = \frac{1}{2\pi i} \int_C e^{\phi z} g(\phi)\, d\phi,$$

where C may be taken to be a circle $|z| = r' > r$. It follows that with $|z| < R$, given, $|G(z)| < K\epsilon$, for some fixed $K > 0$. Since $F^*(z)$ is here an integral function of order at most 1 and minimum type, the uniform convergence to $F(z)$ of (9) is established for finite values of z.

Applying the inverse transform process to (12), we get, with $\lim \alpha_n = \alpha$,

$$F(z) = \sum_{\nu=1}^{\infty} \left\{ \left(\sum_{\mu=1}^{m_\nu} A_{\mu\nu} z^{\mu-1} \right) e^{\alpha_\nu z} - S_\nu(z) e^{\alpha z} \right\} + F^*(z). \tag{14}$$

Here the expression $S_\nu(z)e^{\alpha z}$ removes the same number of initial terms from the expansion in the brackets $\{\}$ obtained after developing as a series in z multiplied by $e^{\alpha z}$ as does $Q_\nu\{(z-\alpha)^{-1}\}$ from the expansion in terms of $(z-\alpha)^{-1}$ in the corresponding bracket of (12).

In (14), $F^*(z)$ is an integral function of at most order 1 and type α, and a similar argument to the above, in which the point $z = \alpha$ plays the part of the origin, is sufficient to establish the uniform convergence of $F(z)$ of (14) for finite values of z.

We thus obtain a classification of integral functions with dominant exponential parts of two kinds: those which are dominated by exponential polynomials and those which are dominated by certain types of Dirichlet series with complex exponents.

The Dirichlet series whose properties were discussed by Regensburger (6) form a special class of those here dealt with. Both satisfy the condition that the $|\alpha_\nu|$ be bounded, but the former also requires the boundedness of the m_ν and the convergence of

$$\sum_{\nu=1}^{\infty} \sum_{\mu=1}^{m_\nu} |A_{\mu\nu}|.$$

5. A generalization

Now suppose that $F(z)$ is an integral function of finite positive order ρ and type h (> 0). Then we can replace (1) by

$$F(z) = \sum_0^\infty a_n z^n \equiv \sum_0^\infty \frac{c_n z^n}{\Gamma(n\sigma+\sigma)}, \tag{15}$$

where $\sigma = \rho^{-1}$. The generalized Laplace transform† of $F(z)$,

$$f(z) = \sum_0^\infty \Gamma(n\sigma+\sigma) a_n z^{-n-1} = \sum_0^\infty c_n z^{-n-1}, \tag{16}$$

is then regular outside the circle $|z| = h^\rho$ and singular at one point of the circumference of this circle at least.

We first note that the generalized inverse transform of

$$(z-\alpha)^{-1} = \sum_0^\infty \alpha^n z^{-n-1}$$

is

$$\sum_0^\infty \frac{\alpha^n z^n}{\Gamma(n\sigma+\sigma)} \equiv E_\rho(\alpha z), \tag{17}$$

so that we wish to obtain necessary and sufficient conditions in order that $F(z)$ may have dominant elements of the form $E_\rho(\alpha z)$, these functions being of order ρ and type $h = |\alpha|^\rho$.

On the other hand, the generalized inverse transform of

$$(z-\alpha)^{-2} = \sum_1^\infty n\alpha^{n-1} z^{-n-1}$$

is

$$\sum_1^\infty \frac{n\alpha^{n-1} z^n}{\Gamma(n\sigma+\sigma)} = \rho \sum_1^\infty \frac{[(n\sigma+\sigma-1)-(\sigma-1)]\alpha^{n-1} z^n}{\Gamma(n\sigma+\sigma)}. \tag{18}$$

† See (3), Part II, where further references will be found.

If ρ is not an integer, other functions of order ρ and type $|\alpha|^\rho$ arise besides $E_\rho(\alpha z)$. If ρ is an integer, then (18) takes the form

$$P(z)+\rho\alpha^\rho z^\rho \sum_\rho^\infty \frac{\alpha^{n-\rho}z^{n-\rho}}{\Gamma[(n-\rho)\sigma+\sigma]}+\frac{\rho-1}{\alpha}\left[E_\rho(\alpha z)-\frac{1}{\Gamma(\sigma)}\right] = Q(z)+E_\rho(\alpha z)(A+Bz^\rho),$$

where $P(z)$, $Q(z)$ are polynomials of degrees $\rho-1$ and A, B are constants.

It is clear that only when ρ is an integer can we obtain a proper generalization of the preceding theory. Even then, in the formula corresponding to (4), the polynomials are functions of z^ρ, and, further, the constants in those polynomials are not the same as the $A_{\mu\nu}$ occurring in the formula corresponding to (3). In fact, the calculations to determine those constants will be extremely heavy and it is not of sufficient interest to undertake them. The fact is that, when $\rho = 1$, then $\rho = \sigma$ and the symmetry of the theory is destroyed when $\rho \neq 1$.

However, there is one case which can be dealt with fairly easily, even when ρ is not an integer, and that is when the poles of $f(z)$ are all simple. We make modifications to Theorem 1 similar to those made in Theorem 2 but resulting from the relations (15)–(17). We write $\mathfrak{D}_{n,s}$ for the determinant

$$|\Gamma[(n+\mu+\nu+1)\sigma]a_{n+\mu+\nu}| \quad (\mu, \nu = 0, 1, 2,..., s),$$

deduced from (5), and $\mathfrak{K}_{n,\nu}(z)$ for the expression obtained by replacing in $K_{n,\nu}(z)$ of (6) every $c_{n+\nu}$ by $\Gamma[(n+\nu+1)\sigma]a_{n+\nu}$ to get the following:

THEOREM 3. *Necessary and sufficient conditions that an integral function of order ρ and mean type $\sum a_n z^n$ shall have dominant elements of the form $E_\rho(\alpha z)$ of type h_1, without polynomial multipliers; secondary elements of the form $E_\rho(\alpha z)$ of type $h_2 < h_1$, without polynomial multipliers, and so on; are that*

$$\limsup_{n\to\infty} |\mathfrak{D}_{n,\nu}|^{1/n} = h_1^{\nu+1} \quad (\nu = 0, 1, 2,..., p_1-2),$$

$$\lim_{n\to\infty} |\mathfrak{D}_{n,p_1-1}|^{1/n} = h_1^{p_1},$$

$$\limsup_{n\to\infty} |\mathfrak{D}_{n,p_1+\nu}|^{1/n} = h_1^{p_1}h_2^{\nu+1} \quad (\nu = 0, 1, 2,..., p_2-2),$$

$$\lim_{n\to\infty} |\mathfrak{D}_{n,p_1+p_2-1}| = h_1^{p_1}h_2^{p_2},$$

and so on; also that the zeros of the polynomials

$$\lim_{n\to\infty} \mathfrak{K}_{n,p_1}(z)/\mathfrak{D}_{n,p_1-1}, \qquad \lim_{n\to\infty} \mathfrak{K}_{n,p_1+p_2}(z)/\mathfrak{D}_{n,p_1+p_2-1},$$

and so on, are all simple. The exponents of type h_1 are given by the zeros α_ν of the first polynomial; the exponents of type h_2 are given by the additional zeros of the second polynomial; and so on.

Since the poles are all simple, $q = p$ and in place of (4) we now have

$$F(z) = \sum_{\nu=1}^{p} A_\nu E_\rho(\alpha_\nu z) + F^*(z), \tag{19}$$

where $F^*(z)$ is an integral function of at most order ρ and type h^* with $h^* < |\alpha_\nu|^\rho$ $(\nu = 1, 2, \ldots, p)$. The constants A_ν are given by (10) after replacing the $c_{n+\nu}$ by $\Gamma[(n+\nu+1)\sigma]a_{n+\nu}$.

A similar classification of integral functions of order ρ and mean type, with dominant elements of the form $E_\rho(\alpha z)$, without polynomial multipliers, holds as in § 4. Corresponding to case (i), $F^*(z)$ in (19) is a polynomial which may reduce to a constant or be zero; in case (ii), $F^*(z)$ is an integral function of at most order ρ and minimum type; in case (iv), $F^*(z)$ is an integral function of at most order ρ and type $h^* < |\alpha_\nu|^\rho$ $(\nu = 1, 2, \ldots, p)$.

The formulae corresponding to (13) and (14) are respectively

$$F(z) = \sum_{\nu=0}^{\infty} [A_\nu E_\rho(\alpha_\nu z) - S_\nu(z)] + F^*(z)$$

and

$$F(z) = \sum_{\nu=0}^{\infty} [A_\nu E_\rho(\alpha_\nu z) - S_\nu(z) E_\rho(\alpha z)] + F^*(z).$$

In the former $\lim \alpha_n = 0$ and $F^*(z)$ is an integral function of order at most ρ and minimum type; in the latter $\lim \alpha_n = \alpha$ and $F^*(z)$ is an integral function of order at most ρ and type α.

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NOTE ON A PERSYMMETRIC DETERMINANT

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It is known [(**1**), (**3**)] that the persymmetric determinant

$$D_n(\lambda, x) = |P_{i+j}^{(\lambda)}(x)/P_{i+j}^{(\lambda)}(1)| \quad (i, j = 0, 1, \ldots, n-1) \tag{1}$$

of order n, where λ is any positive number and $P_n^{(\lambda)}(x)$ denotes the ultraspherical polynomial of degree n and parameter λ, is expressed by means of the relation

$$D_n(\lambda, x) = C_n(\lambda)(x^2-1)^{\frac{1}{2}n(n-1)},$$

where $C_n(\lambda)$ is a constant. Two expressions have recently been obtained for $C_n(\lambda)$. One of these [(**1**) 5] is in the form of a multiple integral:

$$C_n(\lambda) = \frac{1}{n!\,\pi^n} \int_0^{\pi} \cdots \int_0^{\pi} \prod_{q=1}^{n} \sin^{2\lambda-1}\phi_q \prod_{\substack{i,k=1 \\ i<k}}^{n} (\cos\phi_k - \cos\phi_i)^2 \, d\phi_1 \ldots d\phi_n \quad (n \geqslant 2) \tag{2}$$

and the other [(**4**) 151] has the simple explicit form

$$C_n(\lambda) = 2^{n(n-1)} \prod_{r=1}^{n-1} \frac{r!\,\{(\lambda)_r\}^2}{(2\lambda)_{2r}(2\lambda+r-1)_r} \quad (n \geqslant 2), \tag{3}$$

where, as usual, $(a)_r$ denotes $a(a+1)\ldots(a+r-1)$. A determinant closely related to (1) has been considered earlier by MacRobert [(**5**) 4].

The purpose of this note is to show that formula (3) is an almost immediate consequence of a recent result of Brafman's [(**2**) 945]. We use the following special case of one of his formulae:

$$1 + \sum_{n=0}^{\infty} \frac{P_n^{(\lambda)}(0)}{P_n^{(\lambda)}(1)} t^{n+1} = 1 + t\,{}_2F_1(\tfrac{1}{2}, 1; \lambda+\tfrac{1}{2}; -t^2). \tag{4}$$

The method used here seems to have been first applied by Szász (**7**) to the evaluation of persymmetric determinants.

The expression appearing on the right-hand side of (4) has the continued-fraction expansion [(**6**) 348]

$$1 + t\,{}_2F_1(\tfrac{1}{2}, 1; \lambda+\tfrac{1}{2}; -t^2) = 1 + \frac{t}{1-}\,\frac{a_1 t^2}{1-}\,\frac{a_2 t^2}{1-}\cdots,$$

where

$$a_1 = -\frac{1}{2\lambda+1}$$

and

$$a_{2\nu} = -\frac{4\nu(\lambda+\nu-1)}{(2\lambda+4\nu-3)(2\lambda+4\nu-1)}, \qquad a_{2\nu+1} = -\frac{(2\nu+1)(2\lambda+2\nu-1)}{(2\lambda+4\nu-1)(2\lambda+4\nu+1)} \quad (\nu \geqslant 1),$$

which is a special case of Gauss's classical continued-fraction expansion for the hypergeometric function.

Since (2) implies that all the $C_n(\lambda)$ are different from zero, setting

$$\phi_n(\lambda) = (-1)^{\frac{1}{2}n(n-1)}C_n(\lambda),$$

we obtain by a known result in the theory of continued fractions [(6) 324] the relations

$$-\frac{\phi_\nu(\lambda)\phi_{\nu-2}(\lambda)}{\phi^2_{\nu-1}(\lambda)} = \frac{C_\nu(\lambda)C_{\nu-2}(\lambda)}{C^2_{\nu-1}(\lambda)} = -a_{\nu-1} \quad (\nu \geqslant 2),$$

and $\phi_0(\lambda) = C_0(\lambda) = 1$. Since it is directly verified that $C_1(\lambda) = 1$, a simple induction proves (3).

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ON THE SUMMABILITY $|C|$ OF A POWER SERIES

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1. In studying the summability $|C|$ of the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n r^n e^{ni\theta} \tag{1}$$

on its circle of convergence, we may start with a supposition about the function $f(z)$ in the neighbourhood of a point on the circle and about the coefficients. A result of such a kind has been given by the present author,† who proved that, if the radius of convergence of the series (1) is unity and if

$$a_n = O(n^\gamma) \quad (\gamma \geqslant -1), \tag{2}$$

the series (1) is summable $|C, k|$, for every $k > \gamma+1$, at every regular point of $f(z)$ on the unit circle. Since (2) implies

$$\sum_{n=1}^{\infty} n^{-k}|a_n| < \infty, \tag{3}$$

for every $k > \gamma+1$, one may expect that the above result will be still valid if the condition (2) is replaced by the condition (3), and, furthermore, that, if (3) holds, the summability $|C, k|$ of the series (1) at a point on the unit circle depends only on the behaviour of the function $f(z)$ in the neighbourhood of that point. The object of this paper is to prove that these conjectures are certainly true.

It is well known‡ that (3) is itself a necessary condition for the summability $|C, k|$ of the series $\sum a_n e^{ni\theta}$.

2. We suppose throughout that the radius of convergence of the power series (1) is unity. Let $t_n^k(\theta)$, where $k > 0$, denote the nth Cesàro mean of order k of the sequence $na_n e^{ni\theta}$, i.e.

$$A_n^k t_n^k(\theta) = \sum_{\nu=1}^{n} A_{n-\nu}^{k-1} \nu a_\nu e^{\nu i\theta},$$

where

$$A_n^k = \frac{\Gamma(n+k+1)}{\Gamma(n+1)\Gamma(k+1)} \sim \frac{n^k}{\Gamma(k+1)}.$$

† H. C. Chow, *Sci. Record, Acad. Sinica*, (2) 1 (1947), 20–21.

‡ E. Kogbetliantz, *Bull. des Sci. Math.* (2) 49 (1925), 234–56.

The series $\sum a_n e^{ni\theta}$ is said to be *absolutely summable* (C, k), or *summable* $|C, k|$, if its (C, k)-transform $\sum n^{-1} t_n^k(\theta)$ is absolutely convergent.

Now I shall prove the following theorem, which is, in fact, an analogue of a theorem due to A. C. Offord.†

THEOREM 1. *Let θ be a point within an arc (α, β) on the unit circle. If $k > 0$ and $\sum n^{-k}|a_n| < \infty$, the necessary and sufficient condition that the series $\sum a_n e^{ni\theta}$ should be summable $|C, k|$ is that*

$$\sum_{n=1}^{\infty} n^{-1-k} \left| \int_{\alpha}^{\beta} f'(z)\{(e^{i\theta}-z)^{-k}+c_0+c_1 z\} z^{-n}\, dz \right| < \infty, \tag{4}$$

where $|z| = r = 1-1/n$ and $c_0 = c_0(\theta)$ and $c_1 = c_1(\theta)$ are certain constants independent of n.

Proof. Since
$$ze^{i\theta} f'(ze^{i\theta}) = \sum n a_n z^n e^{ni\theta},$$
it follows that
$$\sum_{n=1}^{\infty} A_n^k t_n^k(\theta) z^n = \frac{ze^{i\theta} f'(ze^{i\theta})}{(1-z)^k}.$$

Let C denote the circle $|z| = r < 1$. Then

$$A_n^k t_n^k(\theta) = \frac{1}{2\pi i} \int_C \frac{e^{i\theta} f'(ze^{i\theta})}{z^n (1-z)^k}\, dz = \frac{e^{i(n+k)\theta}}{2\pi i} \int_C \frac{f'(z)}{z^n (e^{i\theta}-z)^k}\, dz. \tag{5}$$

Hence
$$\sum_{n=1}^{\infty} \frac{|t_n^k(\theta)|}{n} = \sum_{n=1}^{\infty} \frac{1}{nA_n^k} \left| \frac{1}{2\pi i} \int_C \frac{f'(z)}{z^n (e^{i\theta}-z)^k}\, dz \right|.$$

Write
$$Q(z) = (e^{i\theta}-z)^{-k}+c_0+c_1 z, \tag{6}$$

where c_0 and c_1 are defined by the equations

$$Q(e^{i\alpha}) = Q(e^{i\beta}) = 0, \tag{7}$$

so that c_0 and c_1 depend on θ but are independent of n. Since

$$\left| \frac{1}{2\pi i} \int_C f'(z)(c_0+c_1 z) z^{-n}\, dz \right| = |na_n c_0+(n-1)a_{n-1} c_1| \tag{8}$$

and $\sum n^{-k}|a_n| < \infty$, it follows that the series $\sum a_n e^{ni\theta}$ is summable $|C, k|$ if and only if

$$\sum_{n=1}^{\infty} n^{-1-k} \left| \int_C f'(z) Q(z) z^{-n}\, dz \right| < \infty.$$

† A. C. Offord, *Proc. London Math. Soc.* (2) 33 (1931), 467–80.

Divide C into two parts: C_1, for which $\alpha < \arg z < \beta$, and C_2. The theorem is proved if we can show that

$$\sum_{n=1}^{\infty} n^{-1-k} \left| \int_{C_2} f'(z) Q(z) z^{-n}\, dz \right| < \infty.$$

Now put $z_1 = re^{i\alpha}$ and $z_2 = re^{i\beta}$. Then, since

$$\frac{Q(z_1)}{z_1 - e^{i\alpha}} \to Q'(e^{i\alpha}) \quad \text{as } r \to 1,$$

we have

$$\left.\begin{aligned} &|Q(z_1)| \leqslant (1-r)\{|Q'(e^{i\alpha})| + K\} = n^{-1}\{|Q'(e^{i\alpha})| + K\}\dagger \\ &\text{and similarly} \\ &|Q(z_2)| \leqslant n^{-1}\{|Q'(e^{i\beta})| + K\} \end{aligned}\right\}. \tag{9}$$

Let z_0 be some point other than the origin inside the circle C. Let

$$F_1(z) - F_1(z_0) = \int_{z_0}^{z} f'(z) z^{-n}\, dz$$

and

$$F_2(z) - F_2(z_0) = \int_{z_0}^{z} F_1(z)\, dz.$$

Then, by partial integration, we have

$$\begin{aligned} I(n) &= \left| \int_{C_2} f'(z) z^{-n} Q(z)\, dz \right| \\ &\leqslant |[F_1(z)Q(z)]_{z_2}^{z_1}| + |[F_2(z)Q'(z)]_{z_2}^{z_1}| + \left| \int_{C_2} F_2(z) Q''(z)\, dz \right|. \end{aligned} \tag{10}$$

Let $r = 1 - 1/n$ and $\mathscr{F}(n) = \sum_{m=1}^{\infty} m|a_m| r^{m-1}$. Then

$$\begin{aligned} \sum_{n=1}^{\infty} n^{-2-k} \mathscr{F}(n) &\leqslant 2 \int_0^1 (1-r)^k\, dr \sum_{m=1}^{\infty} m|a_m| r^{m-1} \\ &= 2 \sum_{m=1}^{\infty} m|a_m| \int_0^1 r^{m-1}(1-r)^k\, dr = 2 \sum_{m=1}^{\infty} |a_m| \frac{\Gamma(m+1)\Gamma(k+1)}{\Gamma(m+k+1)} \\ &\leqslant K \sum_{m=1}^{\infty} m^{-k} |a_m| < \infty. \end{aligned} \tag{11}$$

If z is on C, we have

$$|z^{-n}| = \left(1 - \frac{1}{n}\right)^{-n} < 2e.$$

† I use K to denote a constant independent of n, r, θ, ϕ, whose value may be different from one occurrence to another.

Hence, if z is on C_2,

$$|F_1(z)| \leqslant |F_1(z_0)|+2e|z-z_0| \sum_{m=1}^{\infty} m|a_m|r^{m-1} \leqslant K+K\mathscr{F}(n),$$

and therefore, by (9) and (11),

$$\left.\begin{aligned} \sum_{n=1}^{\infty} n^{-1-k}|F_1(z_1)Q(z_1)| &\leqslant \{|Q'(e^{i\alpha})|+K\} \sum_{n=1}^{\infty} n^{-2-k}|F_1(z_1)| < \infty \\ \sum_{n=1}^{\infty} n^{-1-k}|F_2(z_2)Q(z_2)| &\leqslant \{|Q'(e^{i\beta})|+K\} \sum_{n=1}^{\infty} n^{-2-k}|F_1(z_2)| < \infty \end{aligned}\right\}. \quad (12)$$

Now, since $$F_1(z)-F_1(z_0) = \sum_{m=1}^{\infty} ma_m \int_{z_0}^{z} z^{m-n-1}\,dz,$$

we can write

$$F_1(z) = \sum_{\substack{m=1 \\ m\neq n}}^{\infty} \frac{ma_m}{m-n} z^{m-n}+na_n \log\frac{z}{z_0}$$

and

$$F_2(z) = \sum_{\substack{m=1 \\ m\neq n,n-1}}^{\infty} \frac{ma_m}{(m-n)(m-n+1)} z^{m-n+1}+(n-1)a_{n-1}\log\frac{z}{z_0}+na_n\,g(z),$$

where $g(z) = z\log(z/z_0)-z+z_0$. Write $r = 1-1/n$ and

$$\begin{aligned} |F_2(z)| \leqslant \mathscr{F}_2(n) &= \left\{\sum_{m=1}^{n-2} + \sum_{m=n+1}^{2n} + \sum_{m=2n+1}^{\infty}\right\} \frac{m|a_m|r^{m-n+1}}{|m-n|\,.\,|m-n+1|}+ \\ &\qquad +Kn(|a_{n-1}|+|a_n|) \\ &= \sigma_1(n)+\sigma_2(n)+\sigma_3(n)+Kn(|a_{n-1}|+|a_n|). \end{aligned} \quad (13)$$

Then

$$\begin{aligned} \sum_{n=3}^{\infty} n^{-1-k}\sigma_1(n) &< 2e \sum_{m=1}^{\infty} m|a_m| \sum_{n=m+2}^{\infty} \frac{1}{n^{1+k}|m-n|\,.\,|m-n+1|} \\ &\leqslant 2e \sum_{m=1}^{\infty} m^{-k}|a_m| \sum_{n=2}^{\infty} \frac{1}{n(n+1)} < \infty, \\ \sum_{n=1}^{\infty} n^{-1-k}\sigma_2(n) &< 2e \sum_{n=1}^{\infty} n^{-1-k} \sum_{m=1}^{n} \frac{(m+n)|a_{m+n}|}{m(m+1)} \\ &= 2e \sum_{m=1}^{\infty} \frac{1}{m(m+1)} \sum_{n=m}^{\infty} \frac{(m+n)|a_{m+n}|}{n^{1+k}} \\ &< K \sum_{m=1}^{\infty} \frac{1}{m(m+1)} \sum_{n=m}^{\infty} (m+n)^{-k}|a_{m+n}| < \infty, \end{aligned}$$

$$\sum_{n=1}^{\infty} n^{-1-k}\sigma_3(n) < 2e \sum_{n=1}^{\infty} n^{-1-k} \sum_{m=2n+1}^{\infty} \frac{m|a_m|r^{m-1}}{(m-n)(m-n+1)}$$

$$< 2e \sum_{n=1}^{\infty} n^{-3-k} \sum_{m=1}^{\infty} m|a_m|r^{m-1} < \infty.$$

Hence $$\sum_{n=1}^{\infty} n^{-1-k}\mathscr{F}_2(n) < \infty.$$

Since $Q'(z)$ and $Q''(z)$ are bounded on C_2, we have

$$\sum_{n=1}^{\infty} \frac{|F_2(z)Q'(z)|}{n^{1+k}} \leqslant K \sum_{n=1}^{\infty} n^{-1-k}\mathscr{F}_2(n), \quad \sum_{n=1}^{\infty} \frac{|F_2(z)Q''(z)|}{n^{1+k}} \leqslant K \sum_{n=1}^{\infty} n^{-1-k}\mathscr{F}_2(n). \tag{14}$$

By (10), (12), (14) we have $\sum n^{-1-k}I(n) < \infty$. The theorem is thus proved.

3. It is known that, if the series

$$\sum_{n=1}^{\infty} n^{-1} \int_{\alpha'}^{\beta'} |t_n^k(\theta)|\, d\theta$$

is convergent, the series $\sum n^{-1}|t_n^k(\theta)|$ is convergent for almost all θ in the arc (α', β'). Now, from (5), we have

$$\Sigma = \sum_{n=1}^{\infty} \int_{\alpha'}^{\beta'} \frac{|t_n^k(\theta)|}{n}\, d\theta = \sum_{n=1}^{\infty} \frac{1}{nA_n^k} \int_{\alpha'}^{\beta'} d\theta \left| \frac{1}{2\pi i} \int_C \frac{f'(z)}{z^n(e^{i\theta}-z)^k}\, dz \right|. \tag{15}$$

Let $\alpha < \alpha' < \beta' < \beta$. Write

$$\Sigma \leqslant \Sigma_1 + \Sigma_2 + \Sigma_3, \tag{16}$$

where

$$\Sigma_1 = K \sum_{n=1}^{\infty} \frac{1}{n^{1+k}} \int_{\alpha'}^{\beta'} d\theta \left| \frac{1}{2\pi i} \int_C f'(z)(c_0 + c_1 z)z^{-n}\, dz \right|,$$

$$\Sigma_2 = K \sum_{n=1}^{\infty} \frac{1}{n^{1+k}} \int_{\alpha'}^{\beta'} d\theta \left| \int_{C_2} f'(z)Q(z)z^{-n}\, dz \right|,$$

$$\Sigma_3 = K \sum_{n=1}^{\infty} \frac{1}{n^{1+k}} \int_{\alpha'}^{\beta'} d\theta \left| \int_{C_1} f'(z)Q(z)z^{-n}\, dz \right|.$$

From (6) and (7) we can show easily that $|c_0|$ and $|c_1|$ are each less than

$$H_1(\theta) = |2\sin\tfrac{1}{2}(\beta-\alpha)|^{-1}\{|2\sin\tfrac{1}{2}(\alpha-\theta)|^{-k} + |2\sin\tfrac{1}{2}(\beta-\theta)|^{-k}\}. \tag{17}$$

Hence, by (8),

$$\Sigma_1 \leqslant K\sum_{n=1}^{\infty} n^{-1-k}\int_{\alpha'}^{\beta'} |na_n c_0+(n-1)a_{n-1}c_1|\,d\theta$$

$$\leqslant K\sum_{n=1}^{\infty} n^{-k}\{|a_n|+|a_{n-1}|\}\int_{\alpha'}^{\beta'} H_1(\theta)\,d\theta < \infty.$$

Next, we see from (10) that

$$\Sigma_2 \leqslant K\sum_{n=1}^{\infty} n^{-1-k}\int_{\alpha'}^{\beta'} \left\{|[F_1(z)Q(z)]_{z_2}^{z_1}|+|[F_2(z)Q'(z)]_{z_2}^{z}|+\left|\int_{C_2} F_2(z)Q''(z)\,dz\right|\right\} d\theta.$$

Since

$$|Q'(e^{i\alpha})| = |k(e^{i\theta}-e^{i\alpha})^{-k-1}+c_1| \leqslant k|2\sin\tfrac{1}{2}(\alpha-\theta)|^{-k-1}+H_1(\theta),$$

$$|Q'(e^{i\beta})| = |k(e^{i\theta}-e^{i\beta})^{-k-1}+c_1| \leqslant k|2\sin\tfrac{1}{2}(\beta-\theta)|^{-k-1}+H_1(\theta),$$

it follows from (9) and (12) that

$$\sum_{n=1}^{\infty} n^{-1-k}\int_{\alpha'}^{\beta'} |[F_1(z)Q(z)]_{z_2}^{z_1}|\,d\theta < \infty. \tag{18}$$

If z is on C_2,

$$|Q'(z)| = |k(e^{i\theta}-z)^{-k-1}+c_1|$$

$$\leqslant K\max\{|2\sin\tfrac{1}{2}(\alpha-\theta)|^{-k-1}, |2\sin\tfrac{1}{2}(\beta-\theta)|^{-k-1}\},$$

$$|Q''(z)| = |k(k+1)(e^{i\theta}-z)^{-k-2}|$$

$$\leqslant K\max\{|2\sin\tfrac{1}{2}(\alpha-\theta)|^{-k-2}, |2\sin\tfrac{1}{2}(\alpha-\theta)|^{-k-2}\},$$

so that the integrals of $|Q'(z)|$ and $|Q''(z)|$ over (α',β') are bounded when z is on C_2. Hence, by (14),

$$\sum_{n=1}^{\infty} n^{-1-k}\int_{\alpha'}^{\beta'} |[F_2(z)Q'(z)]_{z_2}^{z_1}|\,d\theta \leqslant K\sum_{n=1}^{\infty} n^{-1-k}\mathscr{F}_2(n) < \infty, \tag{19}$$

and

$$\sum_{n=1}^{\infty} n^{-1-k}\int_{\alpha'}^{\beta'} d\theta\left|\int_{C_2} F_2(z)Q''(z)\,dz\right| \leqslant K\sum_{n=1}^{\infty} n^{-1-k}\mathscr{F}_2(n) < \infty. \tag{20}$$

By (18), (19), (20) we have $\Sigma_2 < \infty$. Hence, by (16), $\Sigma < \infty$ if $\Sigma_3 < \infty$.

Accordingly, we obtain the following theorem:

THEOREM 2. *If $k > 0$ and $\sum n^{-k}|a_n| < \infty$, a sufficient condition that the series $\sum a_n e^{ni\theta}$ should be summable $|C,k|$ for almost all points of the arc (α',β'), where $\alpha < \alpha' < \beta' < \beta$, is that*

$$\sum_{n=1}^{\infty} n^{-1-k}\int_{\alpha'}^{\beta'} d\theta\left|\int_{\alpha}^{\beta} f'(z)\{(e^{i\theta}-z)^{-k}+c_0+c_1 z\}z^{-n}\,dz\right| < \infty,$$

where $|z| = r = 1-1/n$ and $c_0 = c_0(\theta)$ and $c_1 = c_1(\theta)$ are functions of θ defined by (6) *and* (7).

4. To show the applications of Theorems 1 and 2, I prove some theorems in this section.

THEOREM 3. *If* $\sum n^{-k}|a_n| < \infty$, *where* $k > 0$, *and if* $0 \leqslant \gamma < 1$ *and* $f'(z) = O(|e^{i\theta_0}-z|^{-\gamma})$ *in the neighbourhood of the point* $e^{i\theta_0}$, *then the series* $\sum a_n e^{ni\theta_0}$ *is summable* $|C, k|$.

Proof. We may suppose, without loss of any generality, that $\theta_0 = 0$ and that $k+\gamma > 1$. Let $\alpha = -\delta$ and $\beta = \delta$, where δ is any positive number, however small. By Theorem 1, it is sufficient to prove that

$$\sum_{n=1}^{\infty} n^{-1-k}\left|\int_{-\delta}^{\delta} f'(z)\{(1-z)^{-k}+c_0+c_1 z\}z^{-n}\,dz\right| < \infty,$$

where $|z| = r = 1-1/n$. Since

$$\sum_{n=1}^{\infty} n^{-1-k}\left|\int_{-\delta}^{\delta} f'(z)(c_0+c_1 z)z^{-n}\,dz\right|$$

$$\leqslant K\sum_{n=1}^{\infty} n^{-1-k}\int_{-\delta}^{\delta} |1-re^{i\phi}|^{-\gamma}\,d\phi \leqslant K\sum_{n=1}^{\infty} n^{-1-k} < \infty,$$

the theorem is proved if we can show that

$$S = \sum_{n=1}^{\infty} n^{-1-k}\int_{-\delta}^{\delta} |1-re^{i\phi}|^{-k-\gamma}\,d\phi < \infty.$$

Now $$\int_{-\delta}^{\delta} |1-re^{i\phi}|^{-k-\gamma}\,d\phi \leqslant K(1-r)^{1-k-\gamma} = Kn^{k+\gamma-1},$$

since $k+\gamma > 1$. Hence

$$S \leqslant K\sum_{n=1}^{\infty} n^{-2+\gamma} < \infty,$$

since $\gamma < 1$. This proves the theorem.

THEOREM 4. *If* $\sum n^{-k}|a_n| < \infty$, *where* $k > 0$, *if* $|f'(z)| < \chi(\theta)$ *in an arc* (α, β), *where* $\chi(\theta)$ *is integrable in Lebesgue's sense in* (α, β), *and if the function*

$$\Phi(\theta) = \frac{1}{\theta-\theta_0}\int_{\theta_0}^{\theta} \chi(\phi)\,d\phi \quad (\alpha < \theta_0 < \beta)$$

is integrable in Lebesgue's sense in (α, β), *then the series* $\sum a_n e^{ni\theta_0}$ *is summable* $|C, k|$.

Proof. Suppose that $\theta_0 = 0$, so that $\alpha < 0 < \beta$. As in the proof of Theorem 3, we have

$$\sum_{n=1}^{\infty} n^{-1-k} \left| \int_{\alpha}^{\beta} f'(z)(c_0+c_1 z)z^{-n}\, dz \right| \leqslant K \sum_{n=1}^{\infty} n^{-1-k} \int_{\alpha}^{\beta} \chi(\theta)\, d\theta < \infty,$$

and

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1-k} \left| \int_{\alpha}^{\beta} f'(z)(1-z)^{-k} z^{-n}\, dz \right| \\
&\quad \leqslant K \sum_{n=1}^{\infty} \frac{1}{n^{1+k}} \int_{\alpha}^{\beta} \frac{\chi(\theta)}{\{(1-r)^2+\theta^2\}^{\frac{1}{2}k}}\, d\theta = K \sum_{n=1}^{\infty} \frac{1}{n} \int_{\alpha}^{\beta} \frac{\chi(\theta)}{(1+n^2\theta^2)^{\frac{1}{2}k}}\, d\theta \\
&\quad = K \sum_{n=1}^{\infty} \frac{1}{n} \int_{0}^{\beta} \frac{\chi(\theta)}{(1+n^2\theta^2)^{\frac{1}{2}k}}\, d\theta + K \sum_{n=1}^{\infty} \frac{1}{n} \int_{\alpha}^{0} \frac{\chi(\theta)}{(1+n^2\theta^2)^{\frac{1}{2}k}}\, d\theta \\
&\quad = J_1 + J_2 \text{ (say)}.
\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}
\int_{0}^{\beta} \frac{\chi(\theta)}{(1+n^2\theta^2)^{\frac{1}{2}k}}\, d\theta &\leqslant \left[\frac{\theta\Phi(\theta)}{(1+n^2\theta^2)^{\frac{1}{2}k}} \right]_0^{\beta} + K \int_{0}^{\beta} \frac{n^2\theta^2\Phi(\theta)}{(1+n^2\theta^2)^{\frac{1}{2}k+1}}\, d\theta \\
&= Kn^{-k} + K \int_{0}^{\beta} \frac{n^2\theta^2\Phi(\theta)}{(1+n^2\theta^2)^{\frac{1}{2}k+1}}\, d\theta.
\end{aligned}$$

Hence

$$\begin{aligned}
J_1 &\leqslant K \sum_{n=1}^{\infty} n^{-1-k} + K \int_{0}^{\beta} \Phi(\theta) \Big\{ \sum_{n \leqslant \theta^{-1}} n\theta^2 + \sum_{n > \theta^{-1}} n^{-1-k}\theta^{-k} \Big\}\, d\theta \\
&\leqslant K + K < \infty.
\end{aligned}$$

Similarly, we can show that $J_2 < \infty$. The theorem follows from Theorem 1.

THEOREM 5. *If* $\sum n^{-k}|a_n| < \infty$, *where* $k > 0$, *and if*

$$\int_{\alpha}^{\beta} |f'(re^{i\phi})|\, d\phi \leqslant K(1-r)^{-\gamma} \quad \text{as } r \to 1,$$

where $0 \leqslant \gamma < \min(1, k)$, *then the series* $\sum a_n e^{ni\theta}$ *is summable* $|C, k|$ *for almost all points in the arc* (α', β'), *where* $\alpha < \alpha' < \beta' < \beta$.

Proof. By Theorem 2, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{-1-k} \int_{\alpha'}^{\beta'} d\theta \left| \int_{\alpha}^{\beta} f'(z)\{(e^{i\theta}-z)^{-k}+c_0+c_1 z\}z^{-n}\, dz \right| < \infty.$$

By (17) we have

$$\sum_{n=1}^{\infty} n^{-1-k} \int_{\alpha'}^{\beta'} d\theta \left| \int_{\alpha}^{\beta} f'(re^{i\phi})(c_0+c_1 re^{i\phi}) r^{-n} e^{-ni\phi}\, d\phi \right|$$

$$\leqslant K \sum_{n=1}^{\infty} n^{-1-k} \int_{\alpha}^{\beta} |f'(re^{i\phi})|\, d\phi \int_{\alpha'}^{\beta'} H_1(\theta)\, d\theta \leqslant K \sum_{n=1}^{\infty} n^{-1-k+\gamma} < \infty.$$

Since

$$\sum_{n=1}^{\infty} n^{-1-k} \int_{\alpha'}^{\beta'} d\theta \left| \int_{\alpha}^{\beta} f'(z)(e^{i\theta}-z)^{-k} z^{-n}\, dz \right|$$

$$\leqslant K \sum_{n=1}^{\infty} n^{-1-k} \int_{\alpha}^{\beta} |f'(re^{i\phi})|\, d\phi \int_{\alpha'}^{\beta'} |e^{i\theta}-re^{i\phi}|^{-k}\, d\theta$$

$$\leqslant K \sum_{n=1}^{\infty} n^{-1-k} \int_{\alpha}^{\beta} |f'(re^{i\phi})|\, d\phi \int_{0}^{2\pi} |1-re^{i\theta}|^{-k}\, d\theta$$

$$\leqslant K \sum_{n=1}^{\infty} n^{-1-k+\gamma} \int_{0}^{2\pi} |1-re^{i\theta}|^{-k}\, d\theta$$

$$\leqslant \begin{cases} K \sum_{n=1}^{\infty} n^{-2+\gamma} < \infty & (k > 1), \\ K \sum_{n=1}^{\infty} n^{-2+\gamma} \log(n+1) < \infty & (k = 1), \\ K \sum_{n=1}^{\infty} n^{-1-k+\gamma} < \infty & (k < 1), \end{cases}$$

the theorem follows.

5. It is worth while noticing that the condition $\gamma < 1$ in Theorem 3 is necessary. For, let

$$f(z) = \log \frac{1}{1-z} = \sum_{n=1}^{\infty} \frac{z^n}{n}$$

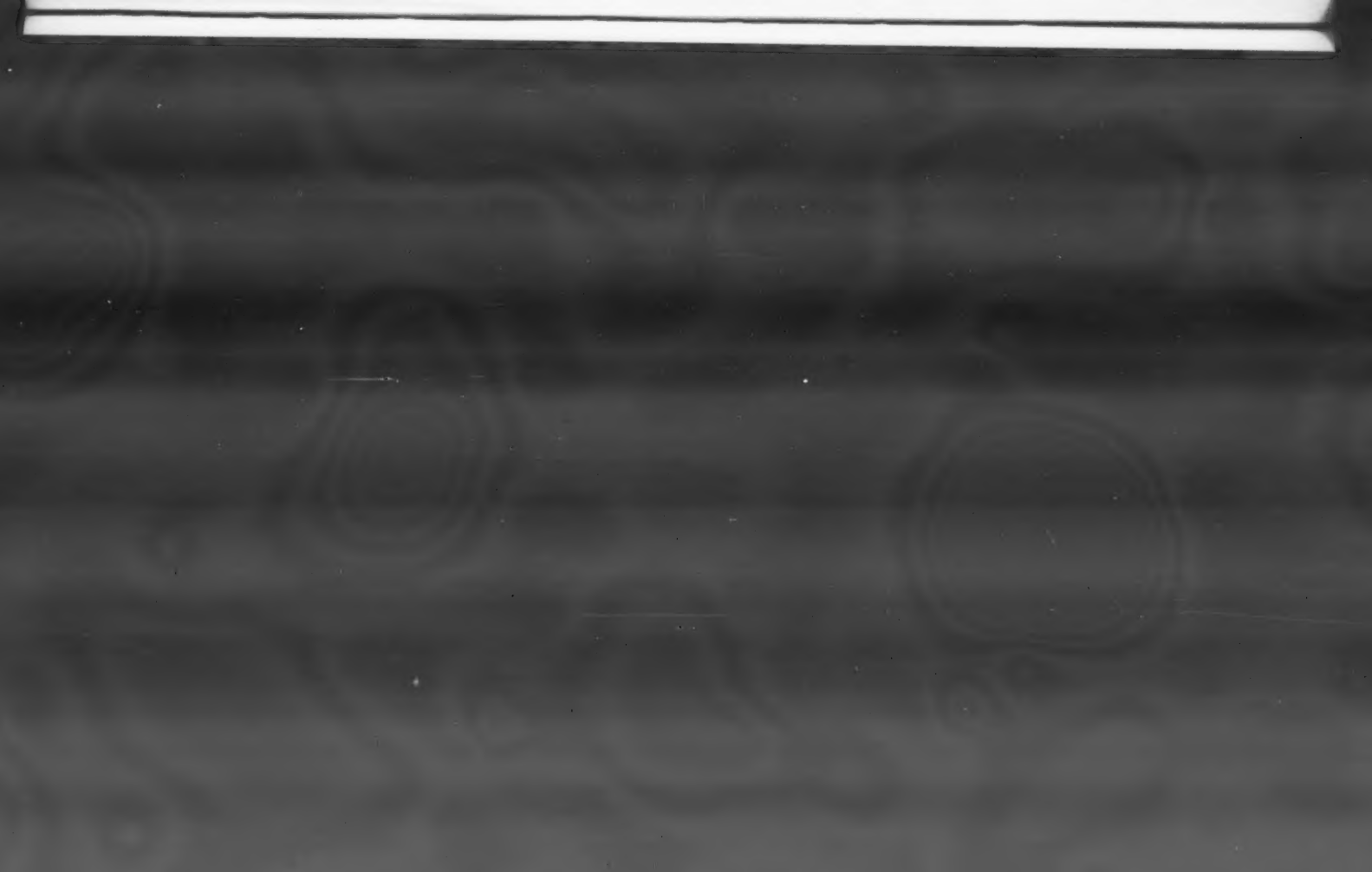
so that

$$f'(z) = (1-z)^{-1}.$$

Then $\sum n^{-k}|a_n| < \infty$ for every $k > 0$ and $|f'(z)| = |1-z|^{-1}$. But the series $\sum n^{-1}z^n$ is not summable $|C, k|$ at the point $z = 1$. On the other hand,

$$\int_{0}^{2\pi} |f'(re^{i\phi})|\, d\phi \leqslant K \log \frac{1}{1-r} \leqslant K(1-r)^{-\gamma},$$

for every $\gamma > 0$. If we choose $\gamma < k < 1$, it follows from Theorem 5 that the series $\sum n^{-1}z^n$ is summable $|C, k|$ for almost all points on the unit circle. In fact, it is summable $|C, k|$ for all points on the unit circle except the point $z = 1$.



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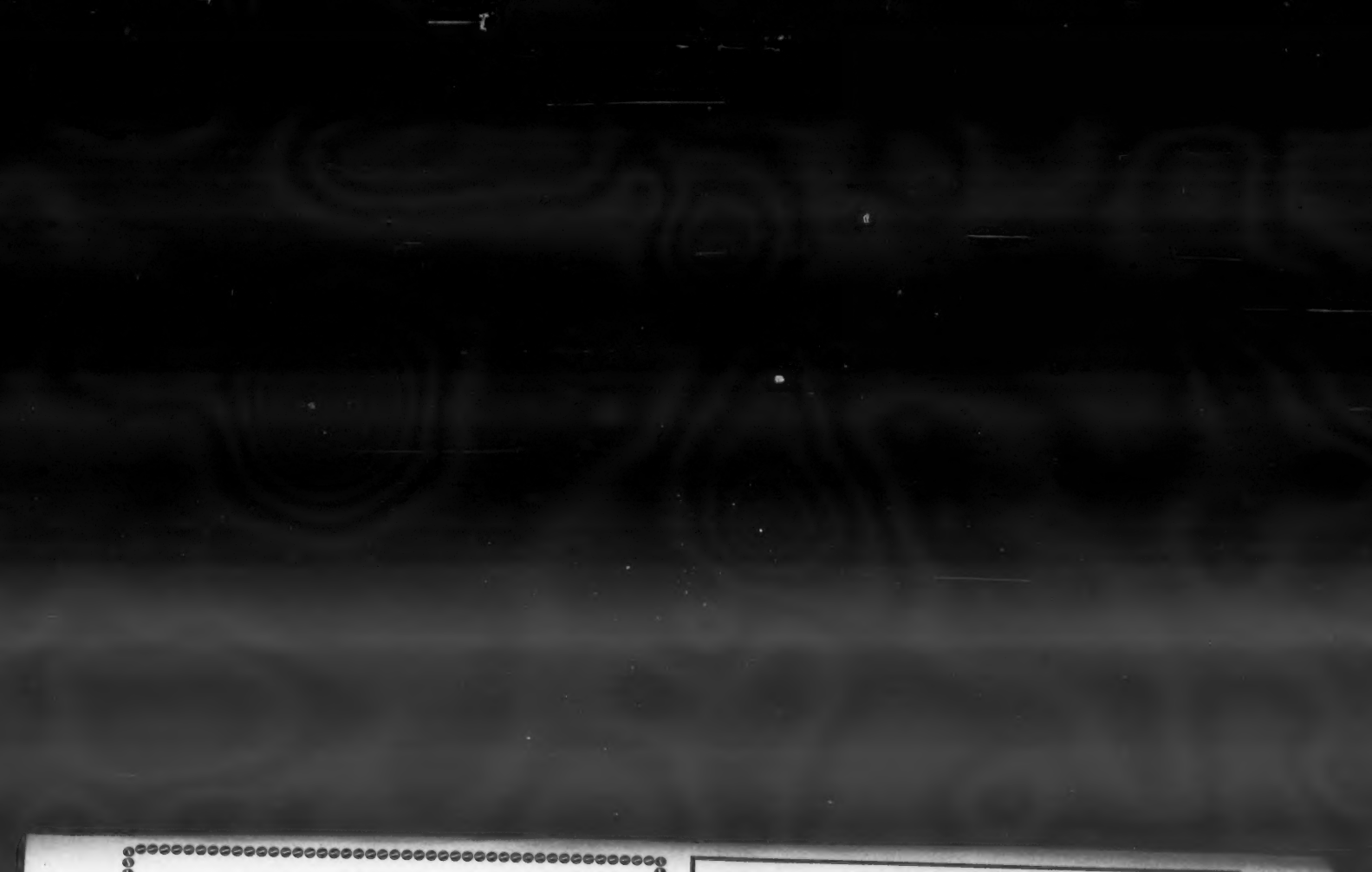
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